- Inverting LaPlace Transforms -

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1 Introduction

These notes are intended to supplement the treatment in the Ogata text. The problem is to recover the function of *time* $f(t)$ given its transform $F(s) = \mathcal{L}[f(t)]$, which is a complex-valued function of a complex variable s. Recall that the complex-valued functions of interest are *rational polynomials*

$$F(s) = \frac{B(s)}{A(s)}$$

where $A(\cdot)$ and $B(\cdot)$ are polynomial functions of s. We assume that A and B have no common factors.

2 The Steps

1. *Reduce to strictly proper form*

   By synthetic division we divide out the whole part and re-write (if necessary) in the form

   $$\frac{B(s)}{A(s)} = \kappa(s) + \frac{\tilde{B}(s)}{A(s)}$$

   where the degree of $\tilde{B}$ is strictly less than that of A.

2. *Factor the demoninator*

   We re-write

   $$A(s) = (s - p_1)\,(s - p_2)\,\ldots\,(s - p_n).$$

Note that if $p_\imath$ is complex (non-zero imaginary part) then its complex conjugate must also appear in the list. For now we assume the roots ($p_\imath$) are distinct.

3. *Expand as a partial fraction*

   We expand the fractional part as a sum of linear factors

$$\frac{\tilde{B}(s)}{A(s)} = \frac{r_1}{(s-p_1)} \; + \; \frac{r_2}{(s-p_2)} \; + \; \ldots + \; \frac{r_n}{(s-p_n)}.$$

4. *Determine the coefficients*

   We find the coefficients from

$$r_\imath = \left[\frac{\tilde{B}(s)\ (s-p_\imath)}{A(s)} \right] \ |_{s=p_\imath}$$

   We can save some work in the case of complex $p_\imath$ by observing that the coefficient associated with the complex conjugate root with be the complex conjugate of the coefficient associated with $p_\imath$. That is, if $p_2 = \bar{p}_1$ then $r_2 = \bar{r}_1$.

5. *Inverse transform term-by-term*

   The $\kappa(s)$ part will produce *generalized* time functions; these are *delta* functions and derivatives. For example,

$$2s^2 + 3s + 4 \to 2\ddot{\delta}(t) + 3\dot{\delta}(t) + 4\delta(t).$$

   The linear factors produce exponentials. For example,

$$\frac{5}{(s-2)} \to 5\exp(2t).$$

   For the complex case we can exploit the conjugate structure

$$\frac{r}{(s-p)} \; + \; \frac{\bar{r}}{(s-\bar{p})} \to 2\mathcal{R}\left[r\exp(pt)\right],$$

   where, $\mathcal{R}[\ldots]$ means *real part of*. For example,

$$\begin{aligned} \frac{-\jmath/4}{(s-2\jmath t)} \; + \; \frac{\jmath/4}{(s+2\jmath t)} \quad &\to \quad 2\mathcal{R}\left[-\jmath/4\exp(2\jmath t)\right] \\ &= \quad 2\mathcal{R}\left[-\jmath/4\left(\cos(2t) + \jmath/4\sin(2t)\right)\right] \\ &= \quad 2\left[(1/4)\sin(2t)\right] \\ &= \quad 1/2\sin(2t). \end{aligned}$$

3 Repeated Roots

In some cases a factor $(s - p_\imath)$ may be repeated; for example $A(s) = (s+1)^2$. In this case we must (slightly ?) modify our formulation and the approach. The partial fraction expansion will be of the form

$$\frac{\tilde{B}(s)}{\tilde{A}(s)(s-p_\imath)^q} = \frac{r_1}{(s-p_\imath)} + \frac{r_2}{(s-p_\imath)^2} + \ldots + \frac{r_q}{(s-p_\imath)^q},$$

where $A(s) = \tilde{A}(s)(s - p_\imath)^q$ so that the repeated root is explicit. The *slick* procedure for computing the coefficients We first write

$$\left[\frac{\tilde{B}(s)\ (s-p_\imath)^q}{A(s)}\right] \Big|_{s=p_\imath} = r_1(s-p_\imath)^{q-1} + r_2(s-p_\imath)^{q-2} + \ \ldots\ + r_{q-1}(s-p_\imath) + r_q.$$

From this we see

$$r_q = \left[\frac{\tilde{B}(s)\ (s-p_\imath)^q}{A(s)}\right] \Big|_{s=p_\imath}.$$

Differentiating both sides leads to

$$r_{q-1} = \frac{d\ \left[\frac{\tilde{B}(s)\ (s-p_\imath)^{q-1}}{A(s)}\right]}{d\ s} \Big|_{s=p_\imath}.$$

The remaining coefficients can be found by continuing up to the $(q-1)$ st derivative. See the example on page 33 of the Ogata book.

The inverse transform of such terms are covered as *item 8* of Table 2-1

$$\frac{1}{(s-p)^q} \rightarrow \frac{1}{(q-1)!} t^{(q-1)}.\exp(pt)$$