Energy-State Models

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April 23, 1998

1 Introduction

The purpose of these notes is to supplement the text material related to energy mangement in atmospheric flight. Energy models provide an alternative to classical climb calculations and are particularly useful for vehicles capable of supersonic flight.

Classical climb focuses on altitude change; that is, in changing the potential energy of the vehicle. Indeed, the analysis includes a force equilibrium requirement which implies *unaccelerated* flight. Since the true airspeed (V) is constant the kinetic energy is likewise, constant.

2 Correcting for Acceleration

It has long been understood that the classical climb analysis has an embedded inconsistency. Recall that we are led to choose a speed V to maximize the rate-of-climb, at a given altitude h. This analysis is repeated at a sequence of altitudes and the resulting family of best speeds defines a function $V^{\mathrm{opt}}(h)$. As the aircraft climbs and the altitude changes, the choice of best speed will vary according to this function. It's clear then that the resulting speed is generally changing with time [$V(t) = V^{\mathrm{opt}}(h(t))$]. Since our choice of speed was based on maximizing the *unaccelerated* rate-of-climb there is an inconsistency. This was well appreciated in the days before WWII and various 'corrections' were suggested. It is useful to consider this issue.

We begin with the equation describing the velocity change from Newton's Laws:

$$m\dot{V} = T - D - W\sin\gamma.$$

Using the $V(t)$ function implied above we are led to compute the time-derivative *via* the chain-rule.

$$m\frac{d\,V}{d\,h}\,\dot{h} = T - D - W\sin\gamma. \tag{1}$$

The result (1) is re-arranged to yield

$$\frac{T-D}{W} = (1/g)\frac{d\,V}{d\,h}\dot{h} + \sin\gamma,$$

and using the kinematic relation $\dot{h} = V \sin\gamma$ we have

$$\frac{T - D}{W} = \left(1 + (V/g)\frac{d\,V}{d\,h}\right)\sin\gamma. \tag{2}$$

Equation (2) is solved for $\sin\gamma$ and used in the kinematic climb-rate expression to produce

$$\dot{h}_a \equiv \frac{(T - D)\,V}{W}\left[1 + (V/g)\frac{d\,V}{d\,h}\right]^{-1}. \tag{3}$$

Result (3) is the rate of climb expression including the effects of acceleration. Note that the first term on the right is our old friend P_s, the specific excess power. In the earlier *unaccelerated climb* analysis we had $\dot{h}_u = P_s$ so that we might also write (3) as

$$\dot{h}_a = \frac{\dot{h}_u}{\left[1 + (V/g)\frac{d\,V}{d\,h}\right]}. \tag{4}$$

This makes it clear that the term in the denominator can be interpreted as a correction applied to the original calculation.

2.1   Climb at Constant EAS

To illustrate these ideas let's suppose that we perform a climb at constant equivalent airspeed (EAS). Since $V = V_e/\sqrt{\sigma(h)}$, it is clear that the airplane will be changing its true airspeed as it climbs. In particular we have

$$\frac{d\,V}{d\,h} = -V_e\sigma^{-3/2}\sigma'/2,$$

so that the correction factor becomes

$$\left[1 + (V/g)\frac{d\,V}{d\,h}\right] = 1 - \left(\frac{V_e^2}{2g}\right)\frac{\sigma'}{\sigma^2}.$$

Since $\sigma' < 0$ the factor is greater than one and the accelerated rate-of-climb is *less* than the unaccelerated prediction. In energy terms it is clear that the aircraft is gaining kinetic energy so that the gain in potential energy is diminished from the earlier estimate.

2.2   Dilemma

While the correction procedure does provide a way to account for the fact that classical climb leads to $V^{\mathrm{opt}}(h)$, it does not completely resolve our problem. Specifically, the velocity profile $V^{\mathrm{opt}}(h)$ was computed based on the best rate-of-climb in *unaccelerated* flight. Can we account for acceleration *before* we do the optimization?

3   Energy Height

The piston-powered aircraft of WWII vintage had limited speed-range so that the amount of stored kinetic energy was generally small compared to the potential energy. For example, a speed change from 100 mph to 400 mph requires an increase in kinetic energy per unit mass of $\Delta(\frac{V^2}{2}) \approx 1.6 \ 10^5 (\text{ft/s})^2$. To achieve the same energy change in potential form requires an altitude increase of about 5000 ft. Thus, a significant change in speed is equivalent, in an energy sense, to a rather modest change in altitude. For this reason it was mostly okay to ignore kinetic energy for such vehicles. Note, however, that the kinetic energy varies as the *square* of the speed; doubling the speed will increase the kinetic energy by a factor of four. As the speed capabilities increase, the kinetic energy becomes increasingly important.

3.1   F. Kaiser and the ME-262

The correction ideas noted above were well-known in the early 1940's as the combatants struggled to develop jet-powered aircraft. The first operational jet-fighter was the Messerschmitt 262 and one of the young flight test engineers at Lager Lechfeld, near Augsburg in southern Germany, was concerned with best-climb predictions. F. Kaiser's [1] [1] idea was to focus on the *total* energy, the sum of kinetic and potential energies:

$$\mathcal{E} = \frac{V^2}{2} \ + \ gh. \tag{5}$$

In this form the energy per unit mass has the dimensions of velocity-squared. Kaiser suggested a normalized form of (5)

$$E \equiv \mathcal{E}/g = \frac{V^2}{2g} \ + \ h, \tag{6}$$

which he called the *resultant height*. It has the physical interpretation of the height one could achieve by exchanging all of the kinetic energy to increase the potential energy. Suppose, for example, that your car, travelling at 55 mph, runs out of fuel at the bottom of a hill. How high can the hill be, such that you can barely coast to the top ? Of course, (6) incorporates the idealization that this energy interchange can be made without loss. This is a key concept in the Kaiser idea - that interchange of energy is *fast* and *lossless*. At any instant the pilot can quickly achieve the balance of energy desired at that time. Upward moves that decrease kinetic energy while increasing the potential are *zoom* maneuvers. Those that go downward to increase the kinetic at the expense of potential energy are *dive* maneuvers. The idealization is that these maneuvers can be done instantly and without loss of energy.

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[1] Similar ideas were later published by Lush [2] in the U.K. and by Rutowski [3] in the U.S. It's not clear if the later authors were aware of Kaiser's work.
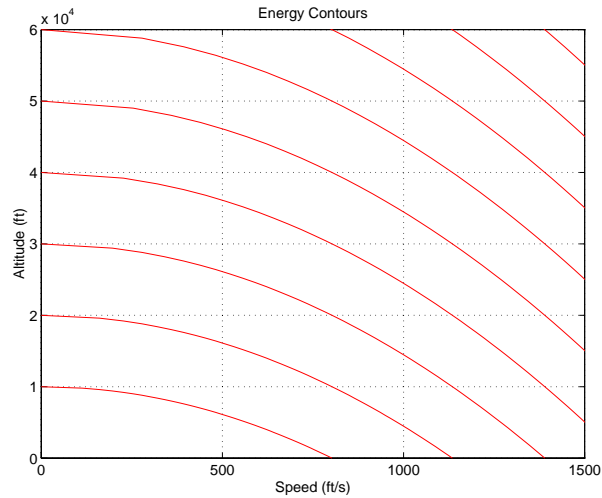
Figure 1: Energy Contours in the Altitude/Airspeed Chart

3.2   Energy Contours

It is helpful to visualize the constant energy lines in the usual altitude-velocity chart. From (6) it is clear that these are downward opening parabolas. They intersect the h axis $(V = 0)$ in an orthogonal way at the value $h = E$. Figure (1), shows some lines of constant energy in the usual altitude-airspeed chart.

3.3   Energy-Rate

While the pilot can re-arrange energy at will, the rate at which the total energy can be changed is finite. To see this we write $\dot{E}$ from (6):

$$\dot{E} = \frac{V \dot{V}}{g} + \dot{h}.$$

We use the usual kinematic result for $\dot{h}$ and the expression (1) for $\dot{V}$ to obtain

$$\dot{E} = (\frac{V}{g})\frac{\left[T - D - W \sin \gamma\right]}{m} + V \sin \gamma.$$

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Note that the $\sin \gamma$ terms cancel and we are left with

$$\dot{E} = \frac{(T - D)V}{W},\tag{7}$$

where we recognize the right-hand side as the specific excess power, P_s ! This, of course, makes perfect sense. In the classical analysis we climb at constant speed so that the energy change is solely through the potential-energy term (*i.e.* the altitude). Thus, our classical climb is a special case. Another special case, sometimes used as a flight-test maneuver, is the level-acceleration wherein the pilot attempts to accelerate at contant altitude. In the ideal *level-accel* the energy change is solely through the kinetic energy term.

3.4   Kaiser Climb

At this point we have apparently come full circle and you may wonder what has been accompished ? In classical climb we have $\dot{h}_u = P_s$ and we are led to maximize P_s. After all this discussion of energy change we have $\dot{E} = P_s$ and we still are led to maximize P_s. So what is the difference ?

At first glance the distinction lies in a seemingly minor change. In the classical case we hold **altitude** fixed and maximize P_s by choice of speed. The Kaiser technique leads us to hold the **energy** constant and to maximize P_s by choice of altitude. Note that since E is fixed one can think of choosing either h or V.

One can easily visualize the distinction between the constant-*altitude* and constant-*energy* procedures by employing the altitude-velocity (h, V) chart. At each point we can compute P_s, where we remember that the drag is calculated such that lift equals weight $(L = W)$. Thus, in the (h, V) chart there will be contours of constant P_s. In the classical climb we look horizontally (*i.e.* at fixed altitude) for the biggest P_s at this altitude. In the Kaiser technique, on the other hand, we look along the energy-parabola appropriate for our current E value.

It turns out that for supersonic aircraft the results of these procedures are remarkably different. For example, in the transonic region there is a significant peak in the drag, largely due to the C_{Do} peak near $M = 1$. The Kaiser approach allows us to *dive* through this region so that it will not choose any points where $M \approx 1$. It may seem somewhat paradoxical; that in order to 'climb' (gain energy) most quickly, the pilot would execute a diving maneuver.

3.5   Numerical Results

To illustrate these ideas we shall display results for a model of the F15-Eagle. The analysis is performed at a fixed weight $(42,000 \text{ lbs})$. The thrust $T(h, M)$ value is generated by interpolating in a table of thrust-values over a grid of altitude and Mach. The drag-polar is parabolic in the lift-coefficient, but the parameters C_{Do} and k are Mach-dependent with values again generated by interpolating over a grid of Mach values.
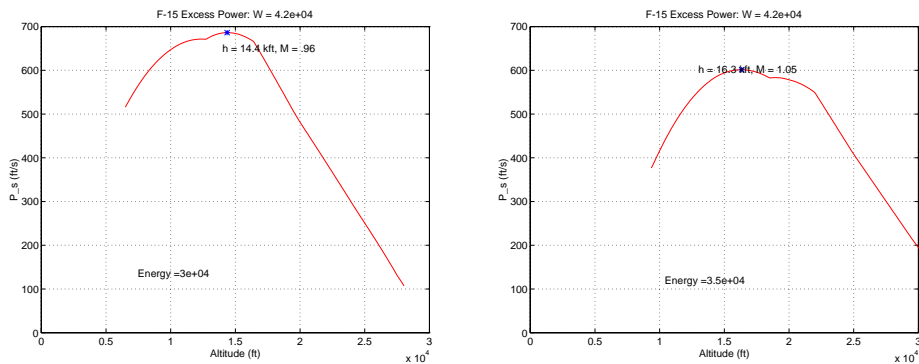
Figure 2: Specific-Excess Power at Two Energy Levels

For a given energy-value we seek to determine the altitude/velocity point such that the specific-excess power is maximized. Recall that [see, equation (7)]

$$P_s = \frac{(T - D)V}{W} = \frac{(P_a - P_r)}{W},$$

where the drag is calculated in level flight $(L = W)$. In the left graph of Figure (2) we show the evaluation of the specific-excess power over a range of altitudes at a fixed energy-value of $E = 30$ kft. Note that the peak occurs at about $h = 14$ kft. This corresponds to a speed of $V \approx 1015$ ft/s or $M \approx .96$. The graph on the right shows results for $E = 35$kft, and we find that $V \approx 1052$ ft/s or $M \approx 1.05$. Note that the global maximizer has shifted from the right local maximizer (subsonic) to the left local maximizer (supersonic). In energy approximation the aircraft can instantly change speed/altitude but the energy must evolve smoothly. This is clearly an approximation to the 'true' behavior but it does hint at some interesting physics, wherein the aircraft uses its potential energy to quickly move through the region of unfavorable transonic drag-rise.

References

[1] F. Kaiser, 'Der Steigflug mit Strahlflugzeugen; Teilbericht 1: Bahnngeschwindigkeit fur Besten Steigens', Versuch-Bericht Nr 262 O2 L 44, Messerschmitt A.G., Lechfeld, May 1, 1944, Translated into English as R.T.P./T.I.B. Translation No. G.D.C./15/148T, Ministry of Supply (UK)

[2] K.J. Lush, 'A Review of the Problem of Choosing a Climb Technique with Proposals for a New Climb Technique for High-Performance Aircraft', Aeronautical Research Council Report Memorandum 2557, 1951.

[3] E.S. Rutowksi, 'Energy Approach to the General Aircraft Performance Problem', Journal of the Aeronautical Sciences, Vol 21, No 3., March 1954.