Clohessy - Wiltshire Analysis

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1 Introduction

Our early analysis of the two-body problem led to a vector (differential) equation of motion in the form

$$\ddot{\vec{r}}(t) = -\frac{\mu}{r(t)^3} \cdot \vec{r}(t) \tag{1}$$

Unfortunately, the general solution to this nonlinear equation does not admit any simple representation. While we were able to establish certain integrals of the motion (for example angular momentum is constant so that the motions are planar) we were forced to abandon the quest for $\vec{r}(t)$ and were led to concentrate on the path equation ($\vec{r}(\nu)$, where ν is the central angle). We made considerable progress, showing that the paths are conic sections and even developing explicit formulae. In order to include time in the analysis, we introduced some auxilliary variables (*the eccentric anomaly*) and while the use of f and g expressions led to useful methods for the Kepler and Lambert problems, we do not have an explicit expression for $\vec{r}(t)$. We now return to the problem of developing such an explicit expression for position as a function of time, but with a simplifying approximation in mind (see [1]).

2 Relative Motions

Consider the situation in the figure. The coordinate system $O - x - y - z$ moves with the origin (O) in a circular path at radius R with the axes so that $\hat{\jmath}_h$ is along the radial direction and $\hat{k}_h$ is orthogonal to the orbit plane. The angular rate of the frame is $\vec{\omega} = \omega \hat{k}_h$ and is related to the radial distance by $\omega^2 R^3 = \mu$.

Consider now an object (satellite) moving in the vicinity of the origin O. Its vector position relative to the central body is $\vec{r}$, while its position relative to the moving frame is $\vec{\rho}$. We have the expected vector relation

$$\vec{r} = \vec{R} + \vec{\rho}. \tag{2}$$

Our basic plan is to use the expression (2) in (1) to obtain an evolution equation for the relative position $\vec{\rho}(t)$. With the approximation that $\rho/R << 1$ we shall obtain an

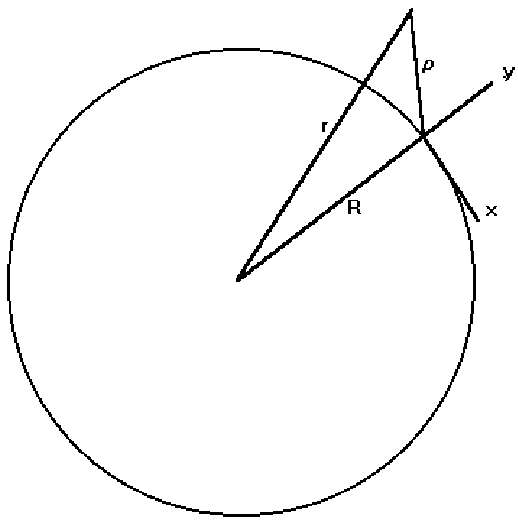

Figure 1: Relative Motion

approximating equation that is *linear*. This is the point of the next section. Before doing so we will generalize (1) to include a forcing term (due to thrust force, or aerodynamic force, or a third gravitating body or *whatever*).

$$\ddot{\vec{r}}(t) = -\frac{\mu}{r(t)^3} \cdot \vec{r}(t) + \vec{f}(t) \tag{3}$$

3 Linearized Model

Substituting (2) in (3) leads to:

$$\ddot{\vec{R}}(t) + \ddot{\vec{\rho}}(t) = -\frac{\mu}{\|\vec{R}(t) + \vec{\rho}(t)\|^3}[\vec{R}(t) + \vec{\rho}(t)] + \vec{f}(t) \tag{4}$$

We begin by considering the scalar denominator term on the *rhs*. For brevity we shall not explicitly display the t-dependence.

$$\begin{aligned}
\|\vec{R} + \vec{\rho}\|^{-3} &= \left([\vec{R} + \vec{\rho}] \cdot [\vec{R} + \vec{\rho}]\right)^{-3/2} \\
&= \left([\vec{R} \cdot \vec{R}] + 2[\vec{R} \cdot \vec{\rho}] + [\vec{\rho} \cdot \vec{\rho}]\right)^{-3/2} \\
&= (1/R^3)\left(1 + \underbrace{\frac{2[\vec{R} \cdot \vec{\rho}]}{R^2} + \frac{[\vec{\rho} \cdot \vec{\rho}]}{R^2}}_{\equiv x}\right)^{-3/2}
\end{aligned}$$

From the Binomial Theorem we have $(1+x)^p = 1 \,+\, px \,+\, \frac{p(p-1)}{2!}x^2 + \ldots$ so that

$$
\begin{aligned}
r^{-3} &= R^{-3}\left(1 + \frac{2(\vec{R}\cdot\vec{\rho})}{R^2} + \frac{(\vec{\rho}\cdot\vec{\rho})}{R^2}\right)^{-3/2} \\
&= R^{-3}\left(1 - \frac{3}{2}\left[2(\vec{R}\cdot\vec{\rho})/R^2 + (\vec{\rho}\cdot\vec{\rho})/R^2\right] + \ldots\right) \\
&= R^{-3}\left(1 - 3\left[\vec{R}\cdot\vec{\rho}\right]/R^2 + \ldots\right)
\end{aligned}
$$

For the *lhs* of equation (4) we use the familiar representation of the (inertial) acceleration as:

$$\ddot{\vec{\rho}}_I = \ddot{\vec{\rho}}_r + 2(\vec{\omega}\times\dot{\vec{\rho}}_I) + \vec{\omega}\times(\vec{\omega}\times\vec{\rho}) + \dot{\vec{\omega}}\times\vec{\rho}. \tag{5}$$

Combining these results our equation (4) becomes

$$
\begin{aligned}
&\ddot{\vec{R}}_I + \ddot{\vec{\rho}}_r + 2(\vec{\omega}\times\dot{\vec{\rho}}_I) + \vec{\omega}\times(\vec{\omega}\times\vec{\rho}) + \dot{\vec{\omega}}\times\vec{\rho} = \\
&\qquad\qquad -\frac{\mu}{R^3}\left(\vec{R} + \vec{\rho} - \frac{3}{R^2}(\vec{R}\cdot\vec{\rho})\vec{R} + \ldots\right) + \vec{f}
\end{aligned} \tag{6}
$$

Since the O moves in a Keplerian orbit we have $\ddot{\vec{R}}_I = -\frac{\mu}{R^3}\vec{R}$ and the model becomes:

$$\ddot{\vec{\rho}}_r + 2\vec{\omega}\times\dot{\vec{\rho}}_r + \vec{\omega}\times(\vec{\omega}\times\vec{\rho}) = -(\frac{\mu}{R^3})\left(\vec{\rho} - \frac{3}{R^2}(\vec{R}\cdot\vec{\rho})\vec{R} + \ldots\right) + \vec{f} \tag{7}$$

As noted in the Figure we choose co–ordinate axes such that

$$\vec{\omega} = \omega\hat{k}_h, \qquad \vec{R} = R\hat{\jmath}_h, \qquad \text{and} \quad \vec{\rho} = x\hat{\imath}_h + y\hat{\jmath}_h + z\hat{k}_h$$

Note that since the origin O is in a *circular* orbit then $\omega^2 R^3 = \mu$. The linear vector equation (7) then leads to the scalar system:

$$
\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= f_x &(8)\\
\ddot{y} + 2\omega\dot{x} - 3\omega^2 y &= f_y &(9)\\
\ddot{z} + \omega^2 z &= f_z &(10)
\end{aligned}
$$

Note that the x and y motions are coupled, and that the z motion is uncoupled from these. Reconcile this with our earlier studies for the (unapproximated) nonlinear model (1).

4 Unforced Motions

We study the unforced case $(\vec{f}(t) \equiv \vec{0})$. The system (8–10) can be 'solved' by a number of methods. Since the z motion is described by (10) alone it's relatively straightforward to find that:

$$z(t) = z_o \cos \omega t + (\frac{\dot{z}_o}{\omega}) \sin \omega t. \tag{11}$$

Equations (8–9) must be analyzed as a coupled system, and this is one of those cases where the characteristic equation has *repeated* roots. After a bit of work we find that:

$$x(t) = x_o \cdot 1 + y_o \cdot 6[\omega t - \sin \omega t] + (\frac{\dot{x}_o}{\omega}) \cdot [4 \sin \omega t - 3\omega t] + (\frac{2\dot{y}_o}{\omega}) \cdot [1 - \cos \omega t] \tag{12}$$

$$y(t) = x_o \cdot 0 + y_o \cdot [4 - 3\cos \omega t] - (\frac{2\dot{x}_o}{\omega}) \cdot [1 - \cos \omega t] + (\frac{\dot{y}_o}{\omega}) \cdot \sin \omega t \tag{13}$$

The coefficients $x_o, \dot{x}_o$, *etc* in (11–13) represent initial values of position and velocity. This linear approximation to the time–position relation is useful in many applications (see, for example [2, 3, 4]).

We have carried out this analysis under the hypothesis that the moving origin O is in a *circular* orbit. The extension to a general Keplerian orbit is not much more complicated.

References

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