Applying the Minimum Principle
Numerical Issues

➤ **Application of the M.P. leads to boundary-value problem(s) including**

- **$2n$ state/adjoint differential**

equations

- extremal control choice from $\min_{u \in \Omega} H$
- initial state/transversality boundary conditions
- final state/transversality boundary conditions

- **possible first-integral from**

$$\frac{d\,H}{dt} = \frac{\partial\,H}{\partial t}.$$

➤ **Numerical solution is accomplished by formulating a Newton root-finding problem, wherein missing state/adjoint boundary values are the**

unknown parameters. Given estimates of these parameters, one can *solve* the initial-value problem and test the specified end-conditions.

- **Several ideas/insights are useful in this task.**

Some observations

Homgeniety of the adjoints

- Our comments focus on the problem with Mayer-cost but can be appropriately extend to the Bolza case.

- **Mulitplying the cost-functional g by a positive constant (say a) amounts to a change in $J-$units and has no effect on the problem.**
- **The adjoint differential equations are linear in the adjoint variables**

- **The optimality condition $\min_{u \in \Omega} H$ is unchanged if all adjoints are multiplied by a positive constant.**
- **If $(\lambda_0, \vec{\lambda}(\cdot))$ leads to an extremal state/control pair $(x^*(\cdot), u^*(\cdot)$, then $(a\lambda_0, a\vec{\lambda}(\cdot))$ leads to the same state/control pair.**

Control Switching/Saturating

- **The optimality condition $\min_{u \in \Omega} H$ often leads to several potential control choices depending on bounds, etc.**
- **Along an extremal arc one**

encounters times at which the control (switching) or its time-derivative (saturation) is discontinuous.

- Numerical IVP solvers commonly use some form of smooth extrapolation, *eg* representing the solution as a

4/5th degree polynomial in t.

- Discontinuities in the control (or its time derivative) in the interior of an integration step degrade solution accuracy
- It's best not to integrate through these points.

State/Adjoint Growth

- Loosely, growth in the (linearized) state system is governed by $\frac{\partial f}{\partial x} \equiv A$
- Similarly, growth in the adjoint system is governed by $-A^T$.

- **If σ is an eigenvalue of the linearized state/adjoint system, then so is $-\sigma$.**
- **Over long integration times the components in the state/adjoint system will differ by many orders of magnitude.**

Minimum-Time Horiztonal Plane Turns

➤ **The terminal transversality conditions for this problem lead**

to

$$\boldsymbol{\lambda}_x(t_f) = \boldsymbol{\lambda}_y(t_f) = 0$$

$$\boldsymbol{\lambda}_u(t_f) = \mu_1, \quad \boldsymbol{\lambda}_\chi(t_f) = \mu_2$$

$$H(t_f) = \mu_1 \dot{u}(t_f) + \mu_2 \dot{\chi}(t_f) = -\boldsymbol{\lambda}_0$$

- **Assuming $\mu_2 \neq 0$ we can use cost-scaling to make**

$$\boldsymbol{\lambda}_\chi(t_f) = \mu_2 = -1$$

Recall we know that $\boldsymbol{\lambda}_\chi < 0$ for a left turn (increasing χ).

- **As a first attempt we can identify a Newton problem with $z \in \mathbb{R}^2$ and identify**

$$z(1) \mapsto \boldsymbol{\lambda}_u(0), \quad z(2) \mapsto t_f$$

- **We are to find $z^* \in \mathbb{R}^2$ so that**

$$u(t_f) = U_f \text{ and } \chi(t_f) = \pi$$

- **If we consider the family of problems parameterized by U_f we can obtain a solution for one-member of the family by choosing $\hat{\lambda}_u(0) < 0$ and**

integrating the IVP forward until $\chi(\hat{t}) = \pi$. Read out the resulting value of $u(\hat{t})$ and the pair $z = (\hat{\lambda}_u(0), \hat{t})$ will solve the problem with $U_f = u(\hat{t})$.