Extending the Minimum Principle State-Dependent Control Constraints

- **Our classes of optimal control problems thus far have all included a specification; $u \in \Omega \subset I\!\!R^m$, Ω a fixed set.**

- **In many applications of interest, this is not sufficiently general. For example, the minimum-time horizontal turns in H.W. Set 6, should incorporate an *aerodynamic* limit on the**

admissible load factor. We have

$$\boldsymbol{n = \frac{L}{W}}$$

where the $\boldsymbol{L}$ is the lift, the component of the aerodynamic force normal to the velocity vector. The usual aerodynamic model expresses the lift in terms

of the *lift-coefficient*

$$C_L = \frac{L}{\bar{q}S},$$

where S is the wing-area and the dynamic-pressure $\bar{q}$ is related to the the air-density ρ and the flight-speed V as

$$\bar{q} = 1/2\rho V^2.$$

- **The aerodynamic-limit imposes an upper-bound on C_L**

$$C_L \leq C_{L\,\max}.$$

In the simplest case, $C_{L\,\max}$ is a specified positive number.

- **In terms of the load-factor n**

the result is

$$n \leq C_{L\,\max} 1/2 \rho V^2 S,$$

so that the limit on the control n depends on the value of state component V.

- **We shall require that the admissible set $\Omega(x) \subset I\!R^m$ is**

specified by a finite number of smooth functions

$$\begin{aligned} \boldsymbol{\beta}_{\imath}(\boldsymbol{x}, \boldsymbol{u}) &= \mathbf{0}, \ \imath = 1, \ldots, \ell \\ \boldsymbol{\beta}_{\ell+\jmath}(\boldsymbol{x}, \boldsymbol{u}) &\geq \mathbf{0}, \ \jmath = 1, \ldots, r \end{aligned}$$

State-Dependent Control Constraints

- **It might seem that admitting $\Omega(x)$ is a minor change in the problem. Actually, it is a major extension.**

- **Consider our original problem**

$$\dot{x} = f(x, u), \text{ with } u \in \Omega I\!\!R^m$$

Let's introduce n additional control variables z and reformulate the problem as

$$\dot{x} = z$$

with the n equality contraints

$$\boldsymbol{\beta(x, u, z) = f(x, u) - z = 0}$$

- **The transformed dynamics appear to be trivial since $\tilde{f}(x, u, z) = z$ is completely independent of the state ! All of the dynamics is hidden in the constraints**

$(u, z) \in \Omega(x) \subset \mathbb{R}^{m+n}$ as represented by the functions $\beta(x, u, z) = 0$.

- Such transformation can be exploited in a finite-dimensional transcription similar to the *POST* and *OTIS* methods discussed earlier.

Extended Minimum Principle State-Dependent Control Constraints

- **In addition to the *variational***

Hamiltonian

$$\begin{aligned} H(\tilde{\lambda}, x, u) &\equiv < \tilde{\lambda}, \tilde{f}(x,u) > \\ &= \Sigma_{i=o}^{n} \lambda_i f_i(x,u) \end{aligned}$$

we define the *augmented*

Hamiltonian

$$\begin{aligned} H^a(\lambda_0, \lambda, \mu, x, u) &= H(\lambda_0, \lambda, \mu, x, u) \\ &+ \Sigma_{\imath=1}^{\ell+r} \mu_\imath \beta_\imath(x,u) \end{aligned}$$

➤ **Note that the $\min H$ operation now requires that the minimization be subject to the constraints**

$$\beta_\imath \geq 0$$

where we have abused the notation since the first ℓ

components of these are equalities.

- **The problem of minimizing H subject to mixed (equality/inequality) constraints is a finite-dimensional problem and is amenable to the treatment in Chapter 3 of**

G-M-W. The K-K-T theory for this problem leads to

- $\boldsymbol{\beta}(x, u^*) \geq 0$, with $\hat{\boldsymbol{\beta}}(x, u^*) = 0$
- $\nabla_u H^a(x, u^*) = 0 \in I\!R^m$
- $\mu_{\ell+\jmath} \geq 0$

➤ **The *Lagrange* multipliers μ for this problem are called**

Valentine **multipliers. Valentine did a 1937(?) thesis at the U. of Chicago under G.A. Bliss.**

- **The adjoint differential equations**

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H}{\partial x}^{T}$$

is replaced by

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial \boldsymbol{H}^{a\,T}}{\partial\, \boldsymbol{x}}$$