Maximum Principle

Theorem: If $x^*(T)$ is on the boundary of the attainable set $K(T)$ and $u^* : [0, T] \mapsto \Omega \subset R^m$ is a corresponding control then there is a non–zero vector–valued

function $\eta(\cdot) : [0, T] \mapsto R^n$ **so that**

$$\dot{\eta}(t) = -A^T(t)\ \eta(t)$$

and

$$\eta^T(t)\ B(t)\ u^*(t) \geq \eta^T(t)\ B(t)\ v$$

for all $v \in \Omega$**.**

The Bushaw Problem

- **Consider the system**

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{b}\boldsymbol{u}(\boldsymbol{t})$$

where

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

- **The control is bounded by**

$$|\boldsymbol{u}(\boldsymbol{t})| \leq \mathbf{1}.$$

- **We seek** $\boldsymbol{K}(\boldsymbol{T}; \mathbf{0}, \mathbf{0})$

Maximum Principle

- **The adjoint system is**

$$\dot{\eta} = -A^T \eta$$

or

$$\begin{aligned} \dot{\eta}_1 &= 0 \\ \dot{\eta}_2 &= -\eta_1 \end{aligned}$$

- $$\eta(t) = \begin{bmatrix} \eta_1^f \\ \eta_2^f + \eta_1^f (T - t) \end{bmatrix}.$$

- $$\begin{aligned} \langle \eta, bu \rangle \; &= \; \langle \begin{pmatrix} \eta_1 \\ \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \rangle u \\ &= \; \eta_2(t) u \end{aligned}$$

- **This implies**

$$
\begin{aligned}
u^*(t) &= +1 \text{ if } \eta_2(t) > 0 \\
u^*(t) &= -1 \text{ if } \eta_2(t) < 0
\end{aligned}
$$

Note that $\eta_2(t) \equiv 0$ implies $\eta_1^f = \eta_2^f = 0$.

- **Since $\eta_2(t)$ is affine, extremal controls switch at most once.**