AOE 5244 - Optimization Techniques

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An Optimal Control Formulation

- Zermelo's Problem
- A Class of Problems: - the Elements
- Generalizations of the Problem

Zermelo's Problem

- **Kinematics**

$$\dot{x}(t) = \kappa y(t) + \cos \beta(t)$$
$$\dot{y}(t) = \sin \beta(t)$$

- **Initial conditions**

$$x(0) = y(0) = 0$$

Zermelo's Problem - cont'd

- **Final conditions**

$$T \text{ specified.}$$

- **Cost functional:** $\min -x(T) \equiv -\int_0^T [\kappa y(t) + \cos\beta(t)] \; dt$

- **We seek** $t \in [0, T] \mapsto \beta(t),\; (i.e.$

an *open-loop* control),

- **where $-\pi < \beta(t) < \pi$;**
- **and, $\beta(\cdot)$ *satisifies* the ODE and the boundary-conditions.**

A Class of Problems: - the Elements

- We have a *dynamic model*

$$\dot{x}(t) = f(x(t), u(t))$$

- where $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is smooth.

- **A control constraint set,**

$$\Omega \subset \mathbb{R}^m$$

$$u(\cdot) : [0, T] \mapsto \Omega \subset \mathbb{R}^m$$

the Elements (cont'd)

- **An initial condition**

  $x(0) = x_o \in \mathbb{R}^n$ **is specified.**

- **A** ***target set*** $\Theta^1 \subset \mathbb{R}^n$

  $\Theta^1 \equiv \{x \in \mathbb{R}^n | \theta_i^1(x) = 0, i = 1, .., q\}$

- **A real-valued cost functional**

$$J(u(\cdot), x(\cdot)) \equiv \int_0^T f_o(x(t), u(t))dt$$

Generalizing the Problem

- A *Bolza* cost functional
  $J(u(\cdot), x(\cdot)) \equiv$
  $\theta_0(x(t_f)) + \int_0^{t_f} f_o(x(t), u(t))dt$
- Explicit time-dependence (in f, θ, *etc.*).

- **State-dependent control bounds**

$$u(t) \in \Omega(x(t)) \subset R^m$$

- **State-inequality constraints**

$$N(x(t)) \leq 0 \in R^p, \quad \forall t.$$

Problems with Special Structure

- ***Slow*** **and** ***Fast*** **states**

$$\begin{aligned} \dot{x}_{\mathrm{s}} &= f_{\mathrm{s}}(x_{\mathrm{s}}, x_{\mathrm{f}}, u) \\ \epsilon \dot{x}_{\mathrm{f}} &= f_{\mathrm{f}}(x_{\mathrm{s}}, x_{\mathrm{f}}, u), \qquad \epsilon > 0 \end{aligned}$$

- **Linearly appearing controls**

(affine)

$$\dot{x} = f(x) + g(x) * u$$

- **Discontinuous state variables (*e.g.* rocket-staging).**

Very Brief History

- **1696 - Brachistochrone Problem**
  Calculus of Variations
- **1919 - Goddard Problem**
  Optimal Control

Brachistochrone

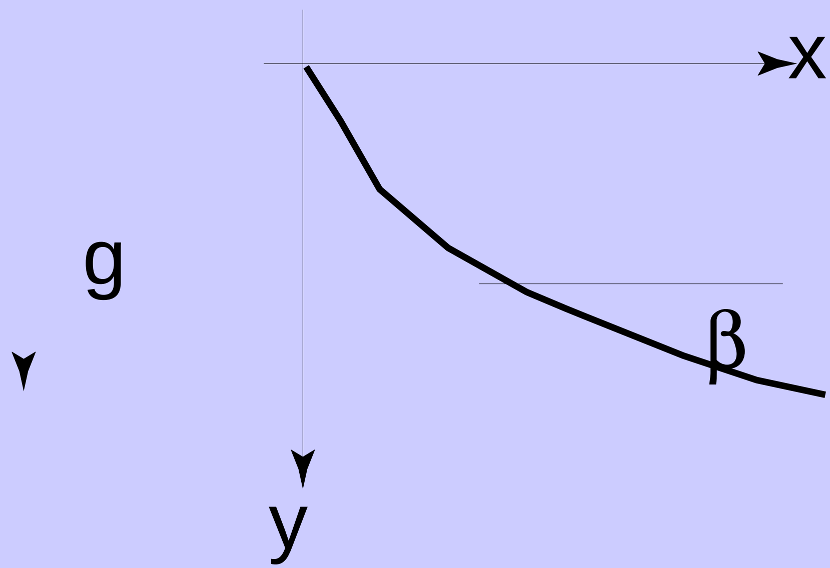

Figure 1: Brachistochrone

Brachistochrone

- From energy

$$E = v^2/2 - g\ y = 0 \rightarrow v = \sqrt{2gy}$$

-

$$\frac{d\ s}{dt} = v = \sqrt{(2gy)}$$

➤ **Solving for dt leads to**

$$T = \int_0^X \sqrt{\frac{1 + y'(x)^2}{2gy(x)}}\, dx$$

Goddard Problem

- Maximize altitude for vertical (only) motion

$$\begin{aligned} \dot{h} &= v \\ \dot{m} &= -\beta \\ \dot{v} &= \left(\beta V_e - D(h, v)\right)/m - g \\ 0 \le \ \beta &\le \beta_{\max} \end{aligned}$$

- **Initial conditions**

$$h(0) = v(0) = 0 \; m(0) = M_o$$

- **Final conditions**

$$M(t_f) = M_f$$

- **Minimize** $-h(t_f)$