Optimal Control - what is a solution ?

- **We seek an *open-loop* function**

$$t \in [0, T] \mapsto u(t) \in \Omega \subset \mathbb{R}^m$$

- **What types of functions are allowed ?**

- **We consider piecewise continuous controls.**

Finding a Solution
two approaches

- Optimize then approximate
- Approximate then optimize

Optimize then Approximiate

C of V - Four Classical Conditions

- **Euler - Lagrange**
- **Legendre**
- **Weierstrass**
- **Jacobi**

Transcribing the Problem
POST Approach

- Program to Optimize Simulated Trajectories
- Finite parameterization of the control function

e.g. **piecewise constant on a specified grid.**

- **Control parameters are available as optimization variables.**
- **States are numerically simulated**

Finite-dimensional Problem

- **Minimize $f(z)$**
- **while satisfying $g(z) = 0$**
- **Nonlinear programming (NLP) problem**

Gill - Murray - Wright Chap 2

Nonlinear Programming (NLP)

➤ Independent variables

$$\mathbf{z} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_n \end{pmatrix} \in \mathcal{R}^n$$

- **Cost Functional**

$$f : \mathcal{R}^n \mapsto \mathcal{R}$$

- **Constraints**

$$g_i(\mathrm{z}) \;=\; 0 \quad i = 1, 2, \ldots, m_e$$

$$g_j(\mathrm{z}) \;\geq\; 0 \quad j = m_e + 1, \ldots, m$$

Special Cases for for Cost Functional

- ➤ single real argument
- ➤ linear function
- ➤ sum of squares of linear functions

- **quadratic function**
- **sum of squares of nonlinear functions**
- **smooth nonlinear function**
- **sparse nonlinear function**
- **non-smooth nonlinear function**

Special Cases for Constraints

- none
- linear function
- simple bounds
- sparse linear functions
- smooth nonlinear functions

- **sparse nonlinear functions**
- **non-smooth nonlinear functions**

Comments

- **Discrete optimization is not considered**
- **Review some linear algebra**

Real (Complex) Vector Space

- **A vector space is a set of elements (vectors) X**
- **vector addition**

$$\boldsymbol{x}, \boldsymbol{y} \in \mathbf{X} \longrightarrow (\boldsymbol{x} + \boldsymbol{y}) \in \mathbf{X}$$

➤ scalar multiplication

$$\boldsymbol{x} \in \mathbf{X}, \boldsymbol{\alpha} \in \mathbb{R} \longrightarrow (\boldsymbol{\alpha}\ \boldsymbol{x}) \in \mathbf{X}$$

Subspaces

- A *subspace* is subset $\mathbf{S} \subset \mathbf{X}$ that is a vector space itself, *i.e.*

$$\mathbf{x}, \mathbf{y} \in \mathbf{S} \longrightarrow (\mathbf{x} + \mathbf{y}) \in \mathbf{S}$$

$$\mathbf{x} \in \mathbf{S}, \boldsymbol{\alpha} \in \mathbb{R} \longrightarrow (\boldsymbol{\alpha}\, \mathbf{x}) \in \mathbf{S}$$

- **In two dimensions the one dimensional subspaces are lines** ***through the origin***
- **$S = \{0\}$ is the zero-dimensional subspace**

Inner-Product Spaces (Hilbert Spaces)

- **An *inner-product* space is a Vector Space X and an inner-product $< x, y >_X$**
- **Given a set S its *orthogonal***

complement **is**

$$\mathbf{S}^{\perp} \equiv \{\mathbf{x} \in \mathbf{X} | < \mathbf{x},\ \mathbf{s} >_{\mathbf{X}} = \mathbf{0} \text{ all } \mathbf{s} \in \mathbf{S}\}$$

- ➤ $\mathbf{S}^{\perp}$ **is a sub-space**
- ➤ **If S is a subspace then**

$$\mathbf{S} \bigoplus \mathbf{S}^{\perp} = \mathbf{X}$$

Special Subspaces for Linear Operators

- Consider a linear map

$$T : \mathrm{X} \mapsto \mathrm{Y}$$

- The *range-space* of T

$$\mathcal{R}(T) \equiv \{\mathrm{y} \in \mathrm{Y} | \mathrm{y} = T\mathrm{x} \text{ some } \mathrm{x} \in \mathrm{X}\}$$

➤ **The *null-space* of T is**

$$\mathcal{N}(T) \equiv \{\mathbf{x} \in \mathbf{X} | T\mathbf{x} = \mathbf{0} \in \mathbf{Y}\}$$

Decomposing Linear Operators

- Based on the *range-space* we can write

$$\mathcal{R}(T) \bigoplus \mathcal{R}(T)^{\perp} = \mathbf{Y}$$

- Based on the *null-space* we can

write

$$\boldsymbol{\mathcal{N}}(\boldsymbol{T}) \bigoplus \boldsymbol{\mathcal{N}}(\boldsymbol{T})^{\perp} = \mathbf{X}$$

Adjoints and Transposes

- **Consider a linear map between inner-product spaces**

$$\boldsymbol{T} : \mathbf{X} \mapsto \mathbf{Y} \qquad \mathbf{X},\ \mathbf{Y}$$

- **For fixed $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{Y}$**

compute

$$< \boldsymbol{T}\mathbf{x},\ \mathbf{y} >_{\mathbf{Y}}$$

- **Suppose we want to do this for lots of x's (fixed** $\boldsymbol{T}$ **and y).**
- **Is there an** $\mathbf{x}^* \in \mathbf{X}$ **so that**

$$< \mathbf{x},\ \mathbf{x}^* >_{\mathbf{X}} = < \boldsymbol{T}\mathbf{x},\ \mathbf{y} >_{\mathbf{Y}} \quad \forall \mathbf{x} \in \mathbf{X}?$$

- **In fact this rule defines the**

adjoint **map**

$$T^* : \mathbf{Y} \mapsto \mathbf{X}$$

- **In terms of the matrix M the calculation looks like:**

$$\begin{aligned} < T\mathrm{x},\ \mathrm{y} >_{\mathrm{Y}} &= < \mathrm{x},\ \mathrm{x}^* >_{\mathrm{X}} \\ (M\,x)^T\,y &= x^T\,(M^T\,y). \end{aligned}$$

Decomposition

- **We have**

$$[\mathcal{R}(T)]^{\perp} = \mathcal{N}(T^*), \quad \text{and}$$
$$\mathcal{R}(T^*) = [\mathcal{N}(T)]^{\perp}.$$

Decomposing Linear Operators

- **Based on the *range-space* we can write**

$$\mathcal{R}(T) \bigoplus \mathcal{R}(T)^{\perp} = \mathbf{Y}$$

- **Based on the *null-space* we can**

write

$$\boldsymbol{\mathcal{N}}(\boldsymbol{T}) \bigoplus \boldsymbol{\mathcal{N}}(\boldsymbol{T})^{\perp} = \mathbf{X}$$

Decomposing Linear Operators

- **Thus, we have**

$$\mathcal{R}(T^*) \bigoplus \mathcal{N}(T)^{\perp} = \mathbf{X}$$

- **and**

$$\mathcal{R}(T) \bigoplus \mathcal{N}(T^*) = \mathbf{Y}$$