Nonlinearly Constrained Problems

$$\min_{\vec{x} \in \mathbb{R}^n} F(\vec{x}) \text{ subject to } c(\vec{x}) = 0 \in \mathbb{R}^m$$

- No complete characterization of the *feasible set*
- *e.g.*

$$
\begin{aligned}
\boldsymbol{c_1(x_1, x_2)} &= \boldsymbol{(x_1 - 1)^2 + x_2^2 - 1} \\
\boldsymbol{c_2(x_1, x_2)} &= \boldsymbol{x_2 - \alpha}
\end{aligned}
$$

Implicit Function Theorem

- **Suppose $f(x^o, y^o) = 0$**
- **Can we *solve* for $y(x)$?**

Implicit Function Theorem

We are given a mapping $f : D \subset R^n \times R^m \mapsto \times R^m$ that is $\mathcal{C}^1$ on the region D, and a point $(x^o, y^o) \in D$ such that $f(x^o, y^o) = 0 \in R^m$. Suppose that

the $(m \times m)$ Jacobian matrix

$$\left[\begin{array}{cccc} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \frac{\partial f_m}{\partial y_2} & \cdots & \frac{\partial f_m}{\partial y_m} \end{array}\right] \Bigg|_{(x_0,\, y_0)}$$

is non-singular. Then there are neighborhoods

S_x of x^o and S_y of y^o, such that for any $\hat{x} \in S_x$ the equation $f(\hat{x}, y) = 0$ has a *unique* solution $y = h(\hat{x}) \in S_y$. Moreover, the mapping $h : S_x \mapsto R^m$ is differentiable with

$$\frac{\partial\, h}{\partial x} = -\left(\frac{\partial\, f}{\partial y}\right)^{-1} \circ \left(\frac{\partial\, f}{\partial x}\right).$$

Constrained Minimization

2-D Case

- Consider $n = 2,\ m = 1.$

$$\min_{(x,y)} F(x, y) \text{ subject to } c(x, y) = 0.$$

- Suppose (x^*, y^*) is a solution

Constrained Minimization

2-D Case

$$f(x,y,\epsilon) \equiv \begin{bmatrix} F(x,y) - F(x^*,y^*) + \epsilon \\ c(x,y) \end{bmatrix}$$

- Note that $f(x^*, y^*, 0) = 0 \in \mathbb{R}^2$
- Suppose (x^*, y^*) is a solution of

the problem and suppose

$$\left[\begin{array}{cc} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} \end{array}\right]_{(x^*, y^*)}$$

has *rank* two.

- **By the *IFT* we can *solve* for x, y in terms of ϵ.**
- **Choose $\epsilon > 0$ and we have a**

point that satisfies the contraints and yields

$$F(x, y) = F(x^*, y^*) - \epsilon < F(x^*, y^*)$$

- **This means (x^*, y^*) is not a solution of the problem.**

➤ **The matrix**

$$\left[\begin{array}{cc} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial c}{\partial x} & \dfrac{\partial c}{\partial y} \end{array}\right]_{(x^*,y^*)}$$

must have rank less than two.

Nonlinearly Constrained Problems

$$\min_{\vec{x} \in \mathbb{R}^n} F(\vec{x}) \text{ subject to } c(\vec{x}) = 0 \in \mathbb{R}^m$$

If $\vec{x}^*$ is a solution, then the matrix

$$\left[\begin{array}{c} \nabla F \\ \nabla c_1 \\ \vdots \\ \nabla c_m \end{array}\right]_{\vec{x}^*}$$

must have rank less than $m + 1$.

Nonlinearly Constrained Problems
Lagrange Muliplier Theorem

$$\min_{\vec{x}\in \mathbb{R}^n} F(\vec{x}) \text{ subject to } c(\vec{x}) = 0 \in \mathbb{R}^m$$

If $\vec{x}^*$ is a solution, then there are scalars $\lambda_0, \lambda_1, \ldots, \lambda_m$, not all zero,

such that

$$\boldsymbol{\lambda}_0 \nabla \boldsymbol{F} + \boldsymbol{\lambda}_1 \nabla \boldsymbol{c}_1 + \ldots + \boldsymbol{\lambda}_m \nabla \boldsymbol{c}_m = \mathbf{0}$$

Nonlinearly Constrained Problems
Lagrange Muliplier Theorem

- Define the *Lagrangian* function

by

$$\mathcal{L}(\vec{x}, \lambda_0, \lambda_1, \ldots, \lambda_m) = \lambda_0 F(\vec{x}) + \sum_{\imath=1}^{m} \lambda_\imath c_\imath(\vec{x})$$

- **If $\vec{x}^*$ is a solution then there is a $\lambda_0, \vec{\lambda}$ so that $\nabla\mathcal{L} = 0$**
- **When can we assume $\lambda_0 \neq 0$?**

- **Suppose that the matrix**

$$\begin{bmatrix} \nabla c_1 \\ \vdots \\ \nabla c_m \end{bmatrix}_{\vec{x}^*}$$

has full rank ($= m$), then

$\lambda_0 \neq 0$.