Lagrange Multipliers and Constraint Qualification

- **For the *linearly constrained* case we have**

$$\nabla F + A^T \lambda = 0 \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^m$$

- **This can be written**

$$\nabla F = A^T(-\lambda)$$

- **No special hypotheses are needed on the linear independence of the columns of A^T (the rows of A).**

Constraint Qualification

- For the nonlinearly constrained problem we get

$$\boldsymbol{\lambda}_0 \nabla F + \boldsymbol{\lambda}_1 \nabla c_1 + \ldots + \boldsymbol{\lambda}_m \nabla c_m = 0$$

- This can be conveniently stated

in terms of a *Lagrangian*

$$\mathcal{L} \equiv \lambda_0 F(x) + \lambda^T c(x)$$

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$$\nabla \mathcal{L} = 0 \in \mathbb{R}^n$$

- **If the set of vectors $\{\nabla c_1, \ldots, \nabla c_m\}$ are linearly independent then $\lambda_0 \neq 0$ and we** can divide by it.

- Such linear indepenence is a *constraint qualification*.
- With the constraint $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the *Jacobian* of c is the $m \times n$ matrix with $\imath$th row $\nabla c_\imath^T$.

Constraint Qualification

- The linear case can be viewed as

$$\boldsymbol{c}(\boldsymbol{x}) \equiv= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$$

- Why is a qualification condition not necessary in the linear case ?

Constrained Qualification

Linear Constraints 3-D Case

- We consider $n = 3, m = 2$
- Each component of $Ax = b$ can be interpeted as $\langle A_\imath, x \rangle = b_\imath$, where $A_\imath$ is the $\imath$-th row of A.

- **Such constraints describe a linear variety (shifted subspace) in $\mathbb{R}^n$.**
- **Generically, these intersect in a line ($n = 3, m = 2$).**
- **If the two rows of A are linearly dependent, then either the**

planes do not intersect, or they are identical

- **in the first case there are no feasible points.**
- **in the second case we have a problem with a single constraint**

Constrained Qualification

Nonlinear Constraints 3-D Case

- We consider $n = 3, m = 2$
- Again, generically we expect that the feasible set is a (curved) line.

- **Suppose**

$$\begin{aligned} c_1 &= (x_1 - 1)^2 + x_2^2 + x_3^2 - 1 \\ c_1 &= x_1 - \alpha \end{aligned}$$

- **Generically, a sphere and a plane intersect in a circle.**

- **If the plane just touches the sphere ($\alpha = 0$) then the**

constraints are linearly dependent (coincident normals).

- **Feasible set is a single point at the origin.**
- **Here $\boldsymbol{\lambda_0 = 0}$.**
- **Intuitively, optimality has nothing to do with $\boldsymbol{F}$.**

Constraint Qualification

- In linearly constrained problems we generically expect $n - m$ degrees of freedom.
- If the constraints are redundant then we have more than this

$n - \text{rank}(A)$.

- In the nonlinear case linear dependence of the constraint normals can lead to fewer than $n - m$ degrees of freedom.

Constraint Qualification
Alternative Form

- **Our Q-C statement has been that the rows of the Jacobian of $\boldsymbol{c}$ ($\boldsymbol{c}_x$, its linearization) are linearly independent.**

- **This means that $\mathcal{N}([c_x]^T) = 0$**
- **Using our decomposition ideas this means**

$$\mathcal{R}([c_x]) = \mathbb{R}^m$$

- **The linearization of c covers all points in the range space.**
- **A point x^* such that**

$\mathcal{R}([c_x]) = I\!R^m$ is a ***regular point*** **for** c.