Trust-Region Methods

- Current values are used to construct a local quadratic model

$$\mathcal{M}_k(p) = F(x_k) + g_k^T p + (1/2) p^T B_k p$$

- **Additionally, we require a trust-region radius, $\boldsymbol{\delta_k}$.**
- **Consider the sub-problem of minimizing $\boldsymbol{\mathcal{M}_k}$, subject to the constraint $\|\boldsymbol{p}\| \leq \boldsymbol{\delta_k}$.**
- **Applying the optimality condition from Chapter 3, the**

solution is given by

$$\boldsymbol{p(\mu) = -\left[B_k + \mu I\right]^{-1} g_k}$$

where μ is selected so that $\|p(\mu)\| = \delta_k$, unless $p(0) \leq \delta_k$, in which case it is the solution.

- Note that $p(0)$ is the Newton step and globally minimizes $\mathcal{M}_k$.

Trust-Region Methods

- **If the Newton step p^{N} is outside the trust region**

$$\delta_k < \|p(0)\| = \| [B_k]^{-1} g_k \|$$

we have to enforce the constraint.

- **The constrained sub-problem of solving for μ such that $\|p(\mu)\| = \delta_k$ is hard. We seek an approximate solution.**
- **The locus of points $p(\mu)$ is a curved line from the Newton step at $\mu = 0$ and approaches a small step along the negative**

gradient $-g_k$ as $\mu \rightarrow \infty$.

- **One approach (M. Powell) is to approximate this curved locus by a piecewise linear one (a sequence of line segments).**
- **The *Cauchy* point p^{CP} is the point along the direction $-g_k$**

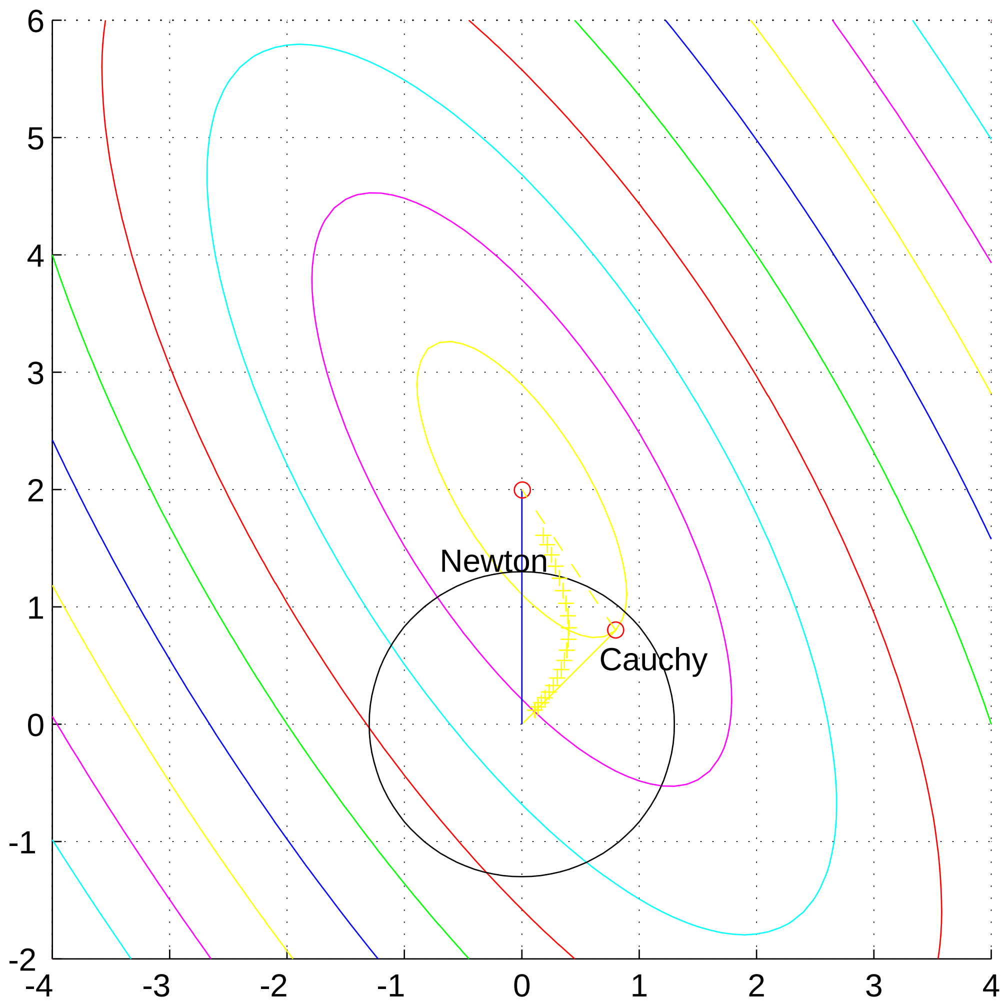
that minimizes $\mathcal{M}_k$. Put

$$p = -\alpha g_k$$

in $\mathcal{M}_k$ and minimize the resulting quadratic function of α.

- The result is

$$\alpha^* = \frac{\|g_k\|^2}{g_k^T B_k g_k}$$


Newton
Cauchy
6
5
4
3
2
1
0
-1
-2
-4
-3
-2
-1
0
1
2
3
4

- **To approximate the locus $p(\mu)$, we use the line segment from the current point to the Cauchy point, followed by the line segment from the Cauchy point to the Newton point. This is the *dog-leg* path.**

- **It can be shown that**

$\|p^{\mathrm{CP}}\| \leq \|p^{\mathrm{N}}\|$. This relies on the fact that $B_k > 0$.

- **Furthermore, it can be shown that the value of the *Model* function monotonically decreases as we move along the piecewise linear dog-leg path.**

- **We approximate the solution to the sub-problem by choosing the point on the dog-leg that is on the boundary of the trust region $\|p\| = \delta_k$.**
- **A *double dog-leg* strategy has been suggested (see *Numerical Methods for Unconstrained***

Optimization and Nonlinear Equations **by J. Dennis and R. Schnabel).**

Trust-Region Methods
Updating the radius

- **Having solved the model sub-problem we must decide whether to accept this step or not. If the step is not accepted then we reduce the size of the**

trust region and re-solve the model problem.

- **Reduction of the trust-region radius is done using the back-tracking strategy.**
- **If the step is accepted and we took the full Newton step, then**

we go to the next (major) iteration. If we have not taken the full Newton step we may wish to increase the trust radius and re-solve the current model problem.

- **This decision is based on comparing the *actual* reduction** in F

$$\Delta F = F(x_+) - F(x_k)$$

and the *predicted* reduction

$$\Delta F_{\text{pred}} = m_q(\alpha^*) - F(x_k)$$

➤ If the prediction is good

$$|\Delta F_{\text{pred}} - \Delta F| < .1|\Delta F|$$

or if the reduction in the

function value is large

$$F(x_+) < F(x_k) + g_k^T(x_+ - x_k)$$

then we re-solve the model problem with $\delta = 2\delta_k$.

If this new trial step does not satisfy the sufficient decrease test, we return to the earlier

point and retain the trust region radius.