Linear Systems
Time-Optimal Control

- **Our system model is**

$$\dot{x}(t) = A(t)\,x(t) \;+\; B(t)\,u(t),$$

with

$x(t) \in R^n$ **and** $u(t) \in \Omega \subset R^m$.

Ω is closed, bounded and convex.

- **Given an initial condition $x(t_o) = x_o \in R^n$ and a (moving) target 'point' $z(t) \in R^n$ we seek a smallest time t^* and a control function $u^*(\cdot)$ so that the solution to DE has $x(t^*) = z(t^*)$.**

Homogeneous System

- **With $u(t) \equiv 0$ we have a homogoeneous system.**
- **The *transition matrix* is a function $\Phi(t, t_o)$ which satisfies**

the matrix differential equation

$$\dot{\Phi}(t, t_o) = A(t)\ \Phi(t, t_o)$$

with initial data $\Phi(t_o, t_o) = I$

- **The transition matrix is a collection of maps which take the initial data to the solution**

at a time t.

$$x(t) = \Phi(t, t_o)x^o$$

solves the homogeneous IVP.

- **In the time-invariant case**

$$\Phi(t, t_o) = F(t - t_o) = \exp[A(t - t_o)],$$

where $\exp[M]$ is the matrix

exponential function

$$\exp[M] = I + M + \frac{M^2}{2!} + \dots$$

- **A pair of matrices from the collection $\{\Phi(t, t_o) | t \geq t_o \in \mathbb{R}\}$ can be combined to get a third matrix. This is the semigroup**

property

$$\Phi(t_2, t_o) = \Phi(t_2, t_1)\ \Phi(t_1, t_o)$$

The picture looks like this:

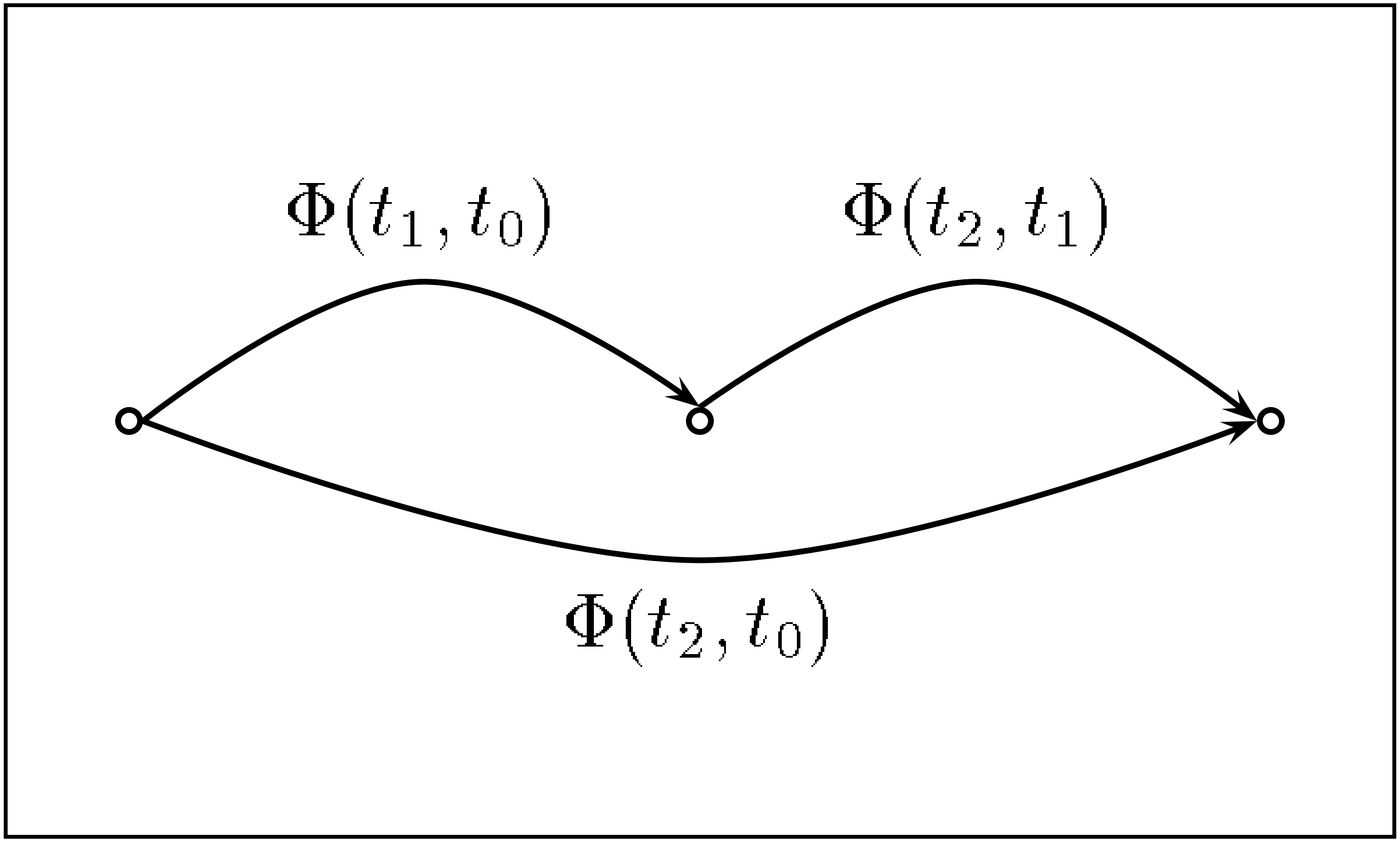

Figure 1: Semigroup Property

Forced System
Variation of Parameters

- **With $u(t) \neq 0$ we have an inhomogoeneous system.**
- **We seek a solution of the form**

$$x(t) = \Phi(t, t_o)\, c(t)$$

- **Substitute into the DE and use properties of Φ to get**

$$\dot{c}(t) = [\Phi(t, t_o)]^{-1} B(t)u(t).$$

- **This leads to the *Variation of***

Parameters formula

$$x(t) = \Phi(t, t_o)\left[x^o + \int_{t_o}^{t} \Phi(t_o, \tau)B(\tau)u(\tau)d\tau\right]$$

Attainable / Reachable Sets

- **Starting at $x^o \in I\!R^n$ at time t_o where can we be at time $t > t_o$?**

$$K(t; t_o, x^o) \equiv \{z \in R^n | z = x(t) \text{ from VP with some } u(\cdot)\}.$$

- **Begin with the simpler set**

$$R(t, t_o) \equiv \{z \in R^n | z = y(t, u) \equiv \int_{to}^{t} \Phi(\tau, t_0)^{-1} B(\tau) u(\tau) d\tau\}$$

- **Comparing the expression for $y(t, u)$ with the VP formula we see that**

$$x(t) = \Phi(t, t_o)[x^0 + y(t, u)].$$

If the set $R(t, t_o)$ is known we can find $K(t; t_o, x^o)$ by translating [adding x^o] and then transforming [by the map $\Phi(t, t_0)$].

- **We can view this as a coordinate transformation $x \leftrightarrow y$. In terms of the y**

coordinates the state equation is

$$\dot{y}(t) = \Phi(t, t_0)^{-1} B(t) u(t)$$

- $K(t; t_o, x^o)$ **is called the** ***attainable set***
- $R(t, t_o)$ **is the** ***reachable set***
- **The sets are closed, bounded and convex.**

- **The optimal control problem can be viewed as seeking the minimum time t^* such that $z(t) \in K(t; t_o, x^o)$**

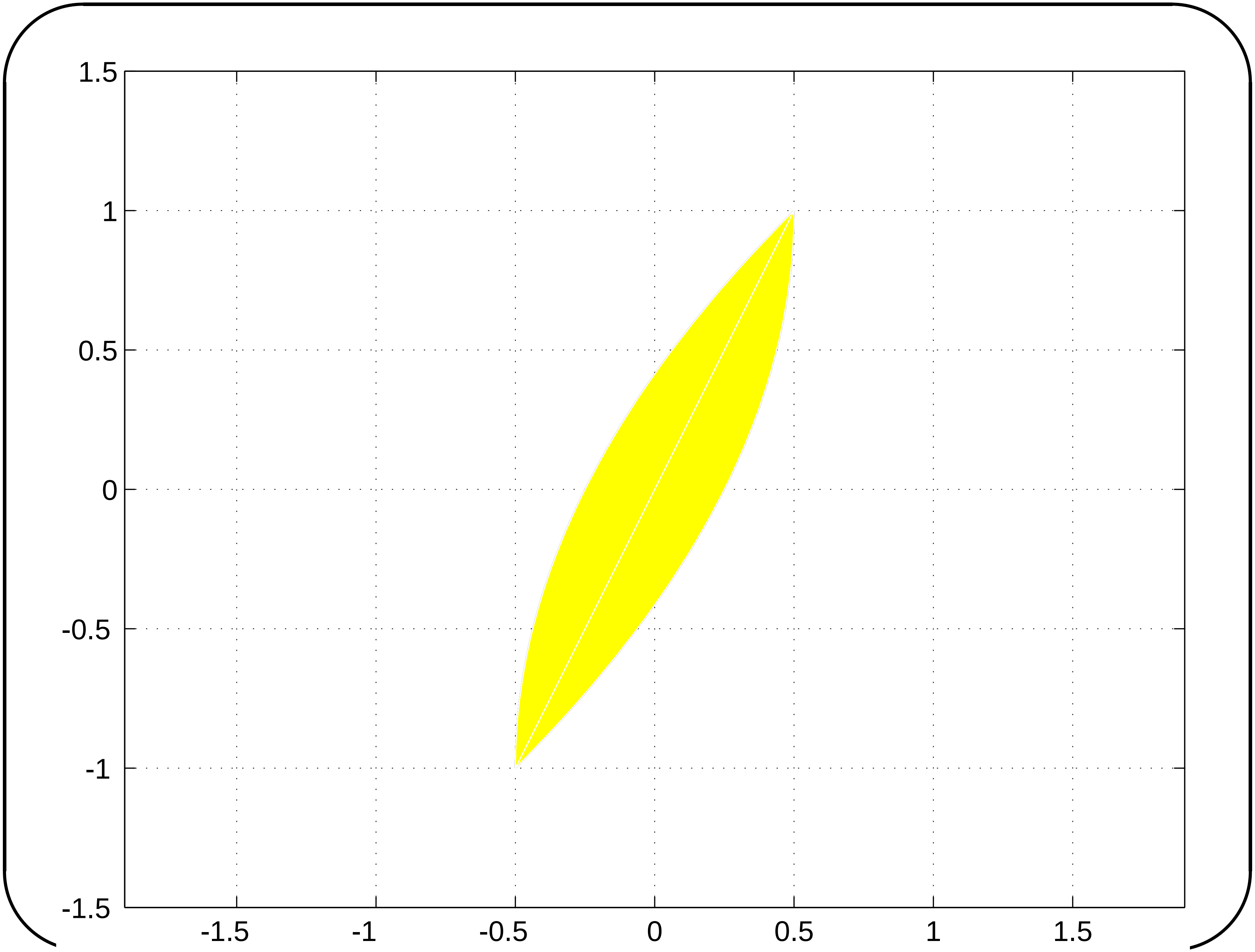

Figure 2: An Attainable Set

Adjoint System

- **Given t_0, t_1 the map $\Phi(t_1, t_o)$ maps initial conditions to solution points**
- **Suppose we want to map a 'plane' of initial points ?**

- **A $n-1$ dimensional plane can be described by $n-1$ linearly independent vectors spanning the plane or by a single vector normal to the plane.**
- **$\Phi(t_1, t_o)$ maps the vectors in the plane. What maps the normals ?**

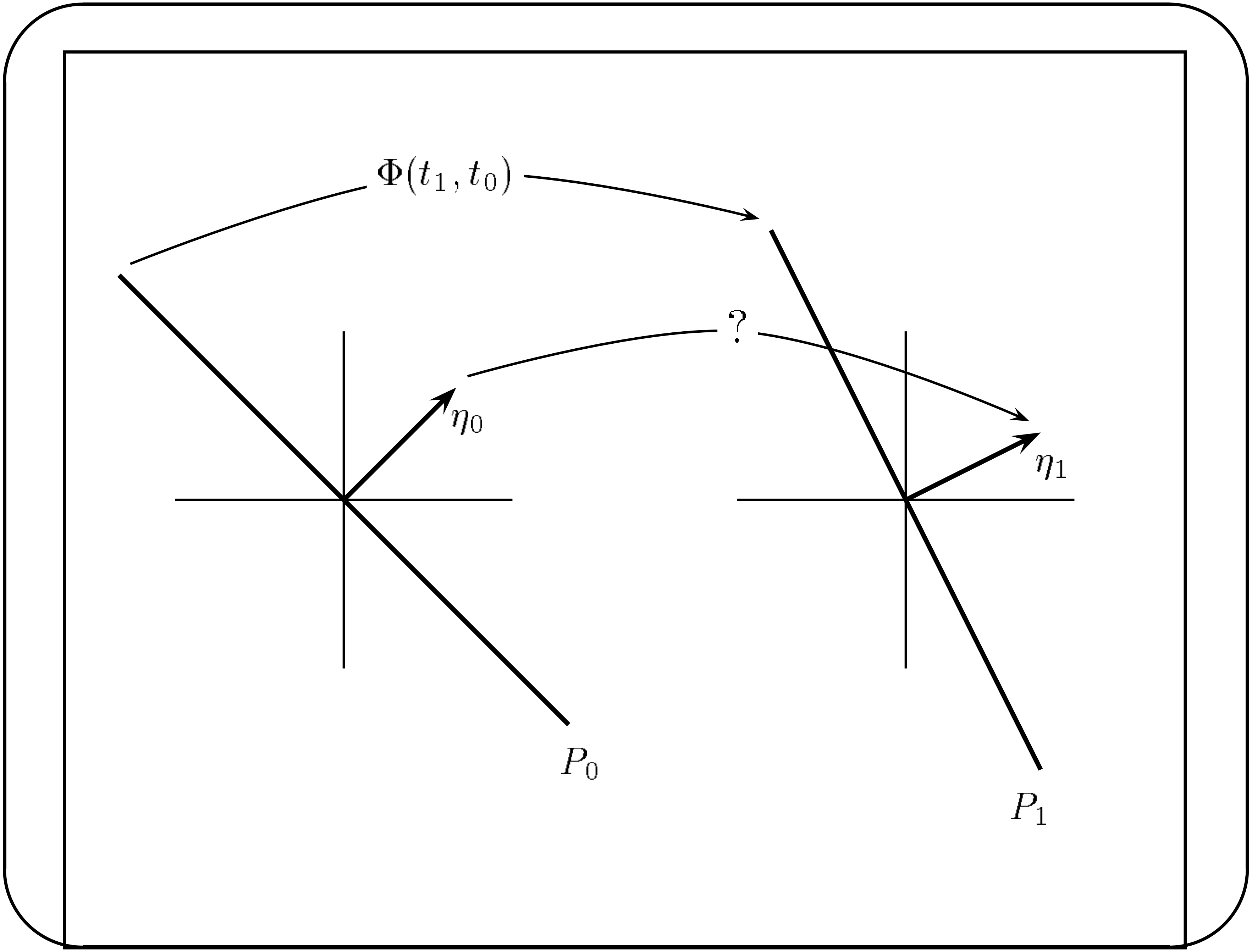

Figure 3: Mapping Planes

Adjoint System - continued

- **Note that $\langle \eta_o, p \rangle = 0$ for all $p \in P_o$ and $\langle \eta_1, p \rangle = 0$ for all $p \in P_1$. Seek $\eta(\cdot)$, so that**

$$\frac{d \, \langle \eta(t), p(t) \rangle}{dt} = 0$$

whenever $p(\cdot)$ satisfies the

homogeneous equation.

- **This leads to**

$$\dot{\eta}(t) = -[A(t)]^T \, \eta(t)$$

this is known as the *adjoint system.*

- **For the inhomogenous system**

$$\frac{d\langle \eta(t), x(t)\rangle}{d\,t} = \langle \eta(t), B(t)u(t)\rangle,$$

so that

$$\langle \eta(t), x(t)\rangle \;=\; \langle \eta(t_o), x(t_o)\rangle + \int_{to}^{t} \langle \eta(\tau), B(\tau)u(\tau)\rangle d\tau.$$

- **If $\Psi(t, t_0)$ is the transition**

matrix for the adjoint system

then

$$[\Psi(t, t_0]^T = [\Phi(t, t_o)]^{-1},$$

- **Our VP formula can be written**

$$x(t) = \Phi(t, t_o)\left[x^o + \int_{to}^{t} [\Psi(t, t_0]^T B(\tau) u(\tau) d\tau\right]$$

Evolution of Reachable Sets

- As t increases the set $R(t, t_o)$ evolves. We want to characterize points on the boundary of the set.
- Suppose that y^* is a point on

the boundary of $R(\hat{t})$ and that $\hat{\eta}$ is an outward normal at this point. If $\tilde{y}$ is any other point in the (the convex set) $R(\hat{t})$ then

$$\langle \hat{\eta}, (\tilde{y} - y^*) \rangle \;\leq\; 0,$$

or

$$\langle \hat{\eta}, y^* \rangle \;\geq\; \langle \hat{\eta}, \tilde{y} \rangle.$$

- **Consider one of these inner-product terms and the**

integral expression for $y(\cdot)$

$$\langle \hat{\eta}, \tilde{y} \rangle$$

$$= \langle \hat{\eta}, \int_{to}^{t} [\Phi(\tau, t_0)]^{-1} B(\tau) u(\tau) d\tau \rangle$$

$$= \int_{to}^{\hat{t}} \langle [\Phi(\tau, t_0)]^{-T} \hat{\eta}, B(\tau) u(\tau) d\tau \rangle$$

$$= \int_{to}^{\hat{t}} \langle \eta(\tau), B(\tau) u(\tau) d\tau \rangle.$$

➤ **Combining these we have**

$$\int_{to}^{\hat{t}} \langle \eta(\tau), B(\tau)u^*(\tau)d\tau \rangle$$

$$\leq \int_{to}^{\hat{t}} \langle \eta(\tau), B(\tau)u(\tau)d\tau \rangle$$

A Maximum Principle

Theorem: If $x^*(T)$ is on the boundary of the attainable set $K(T)$ and $u^* : [0, T] \mapsto \Omega \subset R^m$ is a corresponding control then there is a vector–valued function

$\boldsymbol{\eta(\cdot) : [0, T] \mapsto R^n}$ **so that**

$$\dot{\boldsymbol{\eta}}(\boldsymbol{t}) = -\boldsymbol{A}^T(\boldsymbol{t})\ \boldsymbol{\eta}(\boldsymbol{t})$$

and

$$\boldsymbol{\eta}^T(\boldsymbol{t})\ \boldsymbol{B}(\boldsymbol{t})\ \boldsymbol{u}^*(\boldsymbol{t}) \geq \boldsymbol{\eta}^T(\boldsymbol{t})\ \boldsymbol{B}(\boldsymbol{t})\ \boldsymbol{v}$$

for all $\boldsymbol{v \in \Omega}$**.**