Linear Time-Optimal Control

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1 Motivation

As a first step in studying the modern theory of optimal control we examine the situation wherein the dynamics are described by *linear* ordinary differential equations:

$$\dot{x}(t) = A(t)\, x(t) \ + \ B(t)\, u(t), \tag{1}$$

with $x(t) \in R^n$ and $u(t) \in R^m$. The control values $u(t)$ are restricted to some set $\Omega \subset R^m$. One important case of interest is that of simple bounds $|u_i(t)| \leq 1, \ i = 1, \ldots, m$. More generally, we consider the set Ω to be closed, bounded and convex.

For the optimal control problem we introduce a target 'point' $z(t) \in R^n$. Here we allow $z(\cdot)$ to be a given smooth function of time. One could consider stationary targets but, as we shall see, the moving target introduces no additional complexity. The optimal control problem is to find (for a given initial condition $x(t_o) = x_o \in R^n$, and given target function $z(\cdot)$) a time t^* and a corresponding control function $u^*(\cdot)$ so that the solution to (1) has $x(t^*) = z(t^*)$ and that this is the *minimum time* for which this occurs.

2 Homogeneous System

The study of such (linear, finite-dimensional) dynamical systems begins with the *homogeneous* case $u(t) \equiv 0$. This leads to the study of the *fundamental matrix* (or transition matrix) $\Phi(t, t_o)$ which satisfies the matrix differential equation

$$\dot{\Phi}(t, t_o) = A(t)\ \Phi(t, t_o), \tag{2}$$

with initial data $\Phi(t_o, t_o) = I$, the identity matrix. The transition matrix can also be thought of as a (collection of) map(s) which take the initial data to the solution. That is,

$$x(t) = \Phi(t, t_o) x^o \tag{3}$$

is the solution to

$$\dot{x}(t) = A(t)\ x(t), \tag{4}$$

with initial condition $x(t_o) = x^o$. Note that in general, the matrix-valued function $\Phi(t, t_o)$ has two arguments: t - the current time, and t_o - the initial time. If, however, the matrix A is constant (the time-invariant case) then only elapsed time matters and we write

$$\Phi(t, t_o) = F(t - t_o) = \exp[A(t - t_o)], \tag{5}$$

where $\exp[M]$ is the matrix exponential function

$$\exp[M] = I + M + \frac{M^2}{2!} + \ldots + \frac{M^k}{k!} + \ldots$$

Φ 'inherits' another important property from the differential equation structure. Suppose we start at state x^o at time t_o and propagate the solution forward to time t_1. The resulting state will be, of course, $x(t_1) = \Phi(t_1, t_o)x^o$. Now, use this as an initial state and propagate forward to time t_2. Simple calculations reveal that the state will be $x(t_2) = \Phi(t_2, t_1)[\Phi(t_1, t_o)x^o]$. On the other hand, we could start at x^o and go 'all the way' to t_2 with the expression $x(t_2) = \Phi(t_2, t_o)x^o$. Comparing these two we arrive at the *semigroup* property:

$$\Phi(t_2, t_o) = \Phi(t_2, t_1)\ \Phi(t_1, t_o)$$

The picture looks like this:

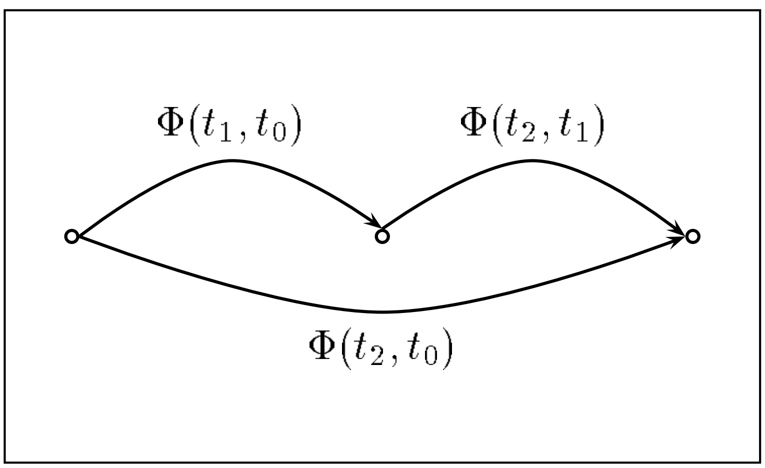

Figure 1: Semigroup Property

The term semigroup indicates that the set of matrices form an *algebraic group* under multiplication. The product of two matrices from the set is another matrix from the set. The *semi*group terminology arise from the fact that, in general, time flows in one direction $t_o < t_1 < t_2$. In the finite-dimensional case we can have the $t_\imath$ in any order.

3 Forced System

Now consider the inhomogeneous system ($u(\cdot) \neq 0$). The idea is to generalize the homogeneous solution (3) by allowing the *constant* x^o to vary; that is we try to find a (vector-valued) function $c(t)$ so that the solution to the inhomogeneous system can be written as

$$x(t) = \Phi(t, t_o)\; c(t).$$

Substituting this form in (1) we have:

$$\dot{\Phi}(t, t_o) c(t) + \Phi(t, t_o)\dot{c}(t) = A(t)\,[\Phi(t, t_o)\; c(t)] \;+\; B(t)u(t).$$

Since $\Phi(t, t_o)$ satisfies the matrix equation (2) we get

$$\dot{c}(t) = [\Phi(t, t_o)]^{-1}\, B(t)u(t).$$

This leads to the *Variation of Parameters* formula

$$x(t) = \Phi(t, t_o)\left[x^o + \int_{to}^{t} [\Phi(\tau, t_o)]^{-1} B(\tau)u(\tau)d\tau\right]. \tag{6}$$

To further justify the formula one notes that

$$\mathrm{DET}[\Phi(t, t_o)] = \exp\left[\int_{to}^{t} \mathrm{Trace}[A(\tau)]d\tau\right],$$

so that $\Phi(t, t_o)$ is never (theoretically) singular.

Based on the semigroup property we write

$$\Phi(t_0, t)\;\Phi(t, t_o) = \Phi(t_o, t_o) = I,$$

so that

$$[\Phi(t, t_o)]^{-1} = \Phi(t_0, t)$$

and (6) can be written

$$x(t) = \Phi(t, t_o)\left[x^o + \int_{to}^{t} \Phi(t_o, \tau)B(\tau)u(\tau)d\tau\right]. \tag{7}$$

4 Attainable / Reachable Sets

Certain results and constructions of optimal control theory are more easily understood in a geometric setting. Given an initial condition $x(t_o) = x^o \in R^n$, we seek the set of points that can be attained at some future time $t > t_o$ using controls in the admissible set Ω.

$$K(t; t_o, x^o) \equiv \{z \in R^n | z = x(t) \text{ from (6) with some } u(\cdot)\}.$$

Note that the control function $u(\cdot)$ must respect the bound Ω. As a first step we study a simpler related set:

$$R(t, t_o) \equiv \{z \in R^n | z = y(t, u) \equiv \int_{to}^{t} [\Phi(\tau, t_0]^{-1} B(\tau) u(\tau) d\tau\} \tag{8}$$

for some admissible control $u(\cdot)$. Comparing with the formula (7) we see that

$$x(t) = \Phi(t, t_o)[x^0 + y(t, u)]. \tag{9}$$

Thus, once the set $R(t, t_o)$ is known we can find $K(t; t_o, x^o)$ by translating [adding x^o] and then transforming [by the map $\Phi(t, t_0)$]. One can view (9) as a change of coordinates from x to y. If this form is substituted into the system-model (1), one finds that:

$$\dot{y}(t) = \Phi(t, t_0)^{-1} B(t) u(t),$$

as expected from the definition of y in (8). $K(t; t_o, x^o)$ is called the *attainable set*, while $R(t, t_o)$ is the *reachable set* (see the book by Hermes and LaSalle, Ref. [1]). Unfortunately, this terminology is not universal. An example of such an attainable set is provided in Fig. (2)

It can be shown that the set $R(t, t_o)$ is closed, bounded and convex (if Ω is). Since $K(t; t_o, x^o)$ is generated from $R(t, t_o)$ by a translation followed by a linear transformation, it also is closed, bounded and convex. It can also be shown that $R(t, t_o)$ varies *continuously* with t. To make sense of this statement, one must describe what it means for two sets to be close. For all these discussions and proofs see [1].

Our optimal control problem can be interpreted as seeking the minimum time t^* such that $z(t) \in K(t; t_o, x^o)$. We can re-formulate this in terms of the reachable set by transforming the target function using the inverse of the map defined in (9)

$$w(t) \equiv [\Phi(t, t_0)]^{-1} z(t) - x_o.$$

Now we seek the minimum time such that $w(t) \in R(t; t_o)$. Note that even if $z(t)$ were constant, the corresponding point $w(t)$ is generally time-varying.

5 Adjoint System

For fixed t_0, t_1 the map $\Phi(t_1, t_o)$ gives the rule by which initial conditions are mapped to solution points (of the homogeneous system). Suppose we have a 'plane' (more precisely an $(n-1)$ dimensional subspace) of 'initial points', say P_o. This will be mapped 'pointwise' by $\Phi(t_1, t_o)$ to another plane at time t_1. Note that alternatively such planes can be described by a single vector, say η_o which is orthogonal to the plane. Rather than describe P_o by specifying $n-1$ vectors in a basis, we specify $[P_o]^\perp$ by giving a single vector η_o. We ask what is the 'map' for the normal vector η_o [see Fig. (3)] ?

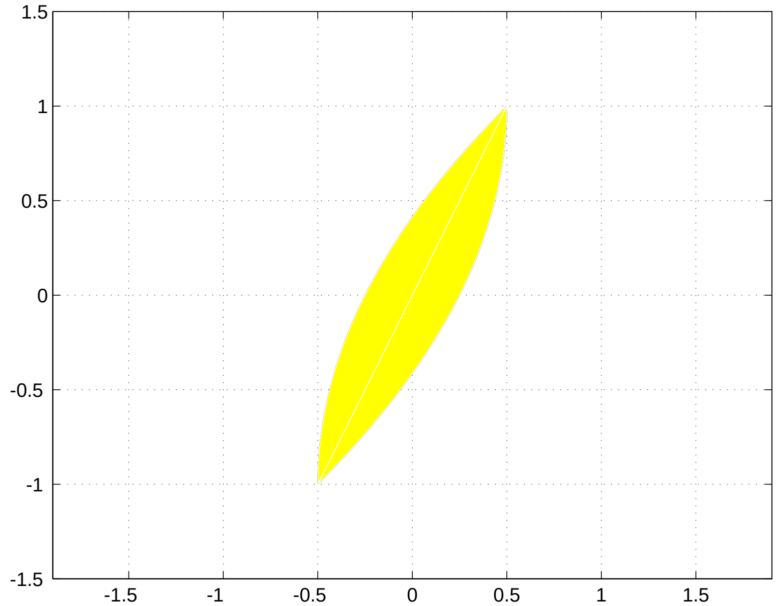

Figure 2: An Attainable Set

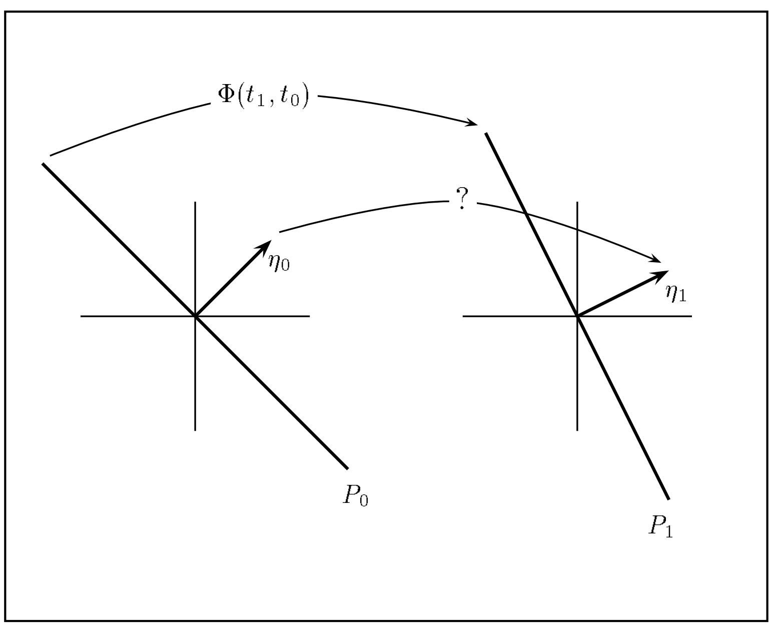

Figure 3: Mapping Planes

Since $< \eta_o, p >= 0$ for all $p \in P_o$ and $< \eta_1, p >= 0$ for all $p \in P_1$, let's try to find $\eta(\cdot)$, so that

$$\frac{d \ < \eta(t), p(t) >}{dt} = 0$$

whenever $p(\cdot)$ satisfies the homogeneous equation (3). Direct calculation shows that

$$\dot{\eta}(t) = -[A(t)]^T \ \eta(t), \tag{10}$$

this is known as the *adjoint system.*

For the inhomogenous system we can directly compute that

$$\frac{d < \eta(t), x(t) >}{d\, t} =< \eta(t), B(t)u(t) >,$$

so that

$$< \eta(t), x(t) > = < \eta(t_o), x(t_o) > + \int_{to}^{t} < \eta(\tau), B(\tau)u(\tau) > d\tau.$$

The adjoint system finds special use in guidance studies wherein one is concerned with the effects of inputs and/or errors on the final state. Let $\Psi(t, t_0)$ be the transition matrix for the adjoint system. We can show that

$$[\Psi(t, t_0]^T = [\Phi(t, t_o)]^{-1},$$

so that (7) can be written as

$$x(t) = \Phi(t, t_o) \left[x^o + \int_{to}^{t} [\Psi(t, t_0]^T B(\tau)u(\tau)d\tau \right]. \tag{11}$$

6 Evolution of Sets

As t increases one envisions the set $R(t)$ evolving; its boundary sweeping through the state space. Points on the boundary are of particular interest.

Suppose that y^* is a point on the boundary of $R(\hat{t})$ and that $\hat{\eta}$ is an outward normal at this point. If $\tilde{y}$ is any other point in the (the convex set) $R(\hat{t})$ then

$$< \hat{\eta}, (\tilde{y} - y^*) > \quad \leq \quad 0,$$

or

$$< \hat{\eta}, y^* > \quad \geq \quad < \hat{\eta}, \tilde{y} > . \tag{12}$$

If we consider one of these inner-product terms and the integral expression for $y(\cdot)$ we find, for example, that

$$\begin{aligned} < \hat{\eta}, \tilde{y} > &= < \hat{\eta}, \int_{to}^{t} [\Phi(\tau, t_0)]^{-1} B(\tau)u(\tau)d\tau > \\ &= \int_{to}^{\hat{t}} < [\Phi(\tau, t_0)]^{-T} \hat{\eta}, B(\tau)u(\tau)d\tau > \\ &= \int_{to}^{\hat{t}} < \eta(\tau), B(\tau)u(\tau)d\tau > . \end{aligned}$$

The last step follows from the property of the adjoint system and $\eta(\cdot)$ is the solution of (10) with boundary condition $\eta(\hat{t}) = \hat{\eta}$. Making use of this result in (12) we have

$$\int_{to}^{\hat{t}} < \eta(\tau), B(\tau)u^*(\tau)d\tau > \quad \leq \quad \int_{to}^{\hat{t}} < \eta(\tau), B(\tau)u(\tau)d\tau > . \qquad (13)$$

For now we specialize to consider the control bound $|u_i(t)| \leq 1$. We have the following result: A point q^* is a boundary point for $R(\hat{t})$ if, and only if, $q^* = y(\hat{t}, u^*)$, where the control u^* is given by

$$u^*(t) = \text{sgn}[\eta^T Y(t)], \qquad (14)$$

where

$$Y(\tau) \equiv [\Phi(\tau, t_o)]^{-1} B(\tau)]$$

and where η is the outward normal to a support plane of $R(\hat{t})$ at the point q^*. In the usual case, the boundary of $R(\hat{t})$ is smooth and the support plane is the tangent plane. More generally, a support plane divides the state space into two 'halves': one 'half' contains the convex set $R(\hat{t})$ and the other 'half' has no points in $R(\hat{t})$.

To further develop the characterization of u^* implied by (14) let's specialize to the case of scalar control so that $Y(\cdot)$ and $B(\cdot)$ are vector-valued (*i.e.* a single column). In this case we have

$$\begin{aligned} < \hat{\eta}, Y(t) > &= < \hat{\eta}, Y(t) > \\ &= < \hat{\eta}, \Phi(t, t_o)^{-1} B(t) > \\ &= < \Phi(t, t_o)^{-T} \hat{\eta}, B(t) > \\ &= < \Psi(t, t_o)\hat{\eta}, B(t) > \\ &= < \eta(t), B(t) > \end{aligned}$$

The first vector in the inner-product is the (time-varying) adjoint vector, a solution to the adjoint system (10). The condition (14) indicates that we choose the control according to the sign of the *switching function* $\sigma(t) \equiv < \eta(t), B(t) >$. For vector-valued control with box constraints we get a switching function for each control component.

7 Summary

Consider the linear dynamical system (1) $u(t) \in \Omega \subset R^m$ (Ω closed, bounded and convex). We are interested in characterizing control histories (*i.e.* functions of time) that lead to points on the boundary of the attainable set $K(t; t_o, x^o)$.

Theorem: If $x^*(T)$ is on the boundary of the attainable set $K(T)$ and $u^* : [0, T] \mapsto \Omega \subset R^m$ is a corresponding control then there is a vector–valued function $\eta(\cdot) : [0,T] \mapsto R^n$ so that

$$\begin{array}{rcl} \dot{\eta}(t) & = & -A^T(t)\ \eta(t) \\ \text{and} & & \\ \eta^T(t)\ B(t)\ u^*(t) & \geq & \eta^T(t)\ B(t)\ v \\ \text{for all} & v \in \Omega. & \end{array}$$

References

[1] Functional Analysis and Time-Optimal Control, Hermes, H. and LaSalle, J.P., Mathematics in Science and Engineering, Vol. 56, Academic Press, 1969.