Mayer Cost Functional

- Our original problem has an integral cost function

$$J(u(\cdot), x(\cdot)) \equiv \int_0^{t_1} f_o(x(t), u(t))dt$$

- Now generalize to include a function of the terminal state

(Mayer - type)

- A real-valued cost functional

$$J(u(\cdot), x(\cdot)) \equiv \int_0^{t_1} f_o(x(t), u(t))dt + g(x(t_1))$$

- This change in the problem statement will result in changes to the Minimum Principle. To

justify these modifications our approach is to transform the new problem (with a Bolza cost functional) to the original form (with a Lagrange cost functional). To carry this our we will augment the state and control vectors.

- **Let**

$$\tilde{x} = \begin{pmatrix} x \\ \tilde{x}_{n+1} \end{pmatrix} \text{ and } \tilde{u} = \begin{pmatrix} u \\ \tilde{u}_{m+1} \end{pmatrix}$$

- **The new state satisfies the differential equation**

$$\dot{\tilde{x}}_{n+1} = \tilde{u}_{m+1}$$

and the initial condition

$\tilde{\boldsymbol{x}}_{n+1}(\mathbf{0}) = \mathbf{0}$. **Note that**

$$\tilde{\boldsymbol{x}}_{n+1}(\boldsymbol{t}_1) = \int_0^{t_1} \tilde{\boldsymbol{u}}_{m+1}(\boldsymbol{t})\, \boldsymbol{dt}$$

- **The new control is unrestricted so we have**

$$\tilde{\Omega} = \Omega \times [-\infty, \infty]$$

- **We also impose the additional**

end-condition

$$\theta^1_{q+1}(\tilde{x}) \equiv \tilde{x}_{n+1} - g(x) = 0$$

➤ **With this construction the cost-functional for the new**

problem can be written as

$$\tilde{J}(\tilde{u}(\cdot), \tilde{x}(\cdot)) \equiv$$

$$\int_0^{t_1} [\underbrace{f_o(x(t), u(t)) + \tilde{u}_{m+1}(t)}_{\tilde{f}_o(\tilde{x}(t), \tilde{u}(t))}]\, dt$$

Analyze the Transformed Problem

- The transformed problem has $n + 1$ state variables and hence $n + 1$ adjoint variables. The variational Hamiltonian can be

written as

$$\tilde{H}(\tilde{\boldsymbol{\lambda}}, \tilde{x}, \tilde{u}) = H(\boldsymbol{\lambda}, x, u) + (\tilde{\boldsymbol{\lambda}}_{n+1} + \boldsymbol{\lambda}_0)\tilde{u}_{m+1}$$

- **It's clear that the adjoint differential equations for the first n components of $\tilde{\boldsymbol{\lambda}}$ are unaltered. The final adjoint**

equation is

$$\dot{\tilde{\boldsymbol{\lambda}}}_{n+1}(t) = 0$$

while the optimality equation is

$$\frac{\partial \tilde{H}}{\partial \tilde{u}_{m+1}} = 0 = (\tilde{\boldsymbol{\lambda}}_{n+1} + \boldsymbol{\lambda}_0)$$

Hence, the new adjoint variable $\tilde{\boldsymbol{\lambda}}_{n+1}(t)$ is constant and equal to $-\boldsymbol{\lambda}_0$.

- **The final transversality condition includes an additional term**

$$\mu_{q+1} \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix}$$

Since this is the only end-condition that depends on

the state-variable $\tilde{x}_{n+1}$ we find that

$$\tilde{\boldsymbol{\lambda}}_{n+1}(\boldsymbol{t}_1) = \boldsymbol{\mu}_{q+1}$$

and hence $\boldsymbol{\mu}_{q+1} = \tilde{\boldsymbol{\lambda}}_{n+1} = -\boldsymbol{\lambda}_0$. Thus, the additional term in the final transversality condition

adds the term

$$\boldsymbol{\lambda}_0\,(\boldsymbol{\nabla g})\,.$$

- **In summary, the terminal transversality condition we had earlier is replaced with Condition (c)':**
  There exists scalars

$\boldsymbol{\mu_i}$, $\quad \boldsymbol{i = 1, \ldots, q}$ **such that:**

$$\boldsymbol{\lambda}(t_1) = \boldsymbol{\lambda}_0 \nabla g(x) +$$

$$\mu_1 \nabla \theta_1(x) + \ldots \mu_q \nabla \theta_q(x)$$