An Example
Zermelo's Problem

- **Our current version of the M.P. admits only problems wherein the end condition are in the form $\theta_i^1(x) = 0$. Since our Zermelo problem has a specified**

final time we include an additional state $x_3 \equiv \tau$

$$
\begin{aligned}
\dot{x} &= \cos\theta + \kappa y \\
\dot{y} &= \sin\theta \\
\dot{\tau} &= 1
\end{aligned}
$$

with $-\pi \leq \theta \leq \pi$.

- **We are to minimize**

$$J[x(\cdot), u(\cdot)] = \int -[\cos\theta + \kappa y] \; dt$$

- **with**

$$\tau(t_f) - T = 0$$

Zermelo Problem

- The variational Hamiltonian is

$$H = (\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_0)\left[\cos\boldsymbol{\theta} + \boldsymbol{\kappa y}\right] + \boldsymbol{\lambda}_y \sin\boldsymbol{\theta} + \boldsymbol{\lambda}_{\boldsymbol{\tau}}$$

- The adjoint differential

equations are

$$\dot{\boldsymbol{\lambda}}_x = 0$$

$$\dot{\boldsymbol{\lambda}}_y = -(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_0)\boldsymbol{\kappa}$$

$$\dot{\boldsymbol{\lambda}}_\tau = 0$$

- **The terminal transversalty**

condition is

$$\vec{\boldsymbol{\lambda}}(t_f) = \mu \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or

$$\begin{aligned} \boldsymbol{\lambda}_x(t_f) &= 0 \\ \boldsymbol{\lambda}_y(t_f) &= 0 \end{aligned}$$

- **This leads to**

$$\boldsymbol{\lambda}_y(t) = \boldsymbol{\lambda}_0 \boldsymbol{\kappa}(t - T)$$

- **From $\min_\theta H$ we get**

$$-(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_0)\sin\theta + \boldsymbol{\lambda}_y \cos\theta = 0$$

or (careful about quadrant here)

$$\tan\theta = \frac{\boldsymbol{\lambda}_y}{(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_0)} = \boldsymbol{\kappa}(T - t)$$

- **Note the M.P. allows $\boldsymbol{\lambda}_0 \geq 0$. If $\boldsymbol{\lambda}_0 = 0$ then we get $\boldsymbol{\lambda}_y = 0$. We already have $\boldsymbol{\lambda}_x = 0$ and the condition $H = 0$ then implies $\boldsymbol{\lambda}_\tau = 0$. Since we know that $(\boldsymbol{\lambda}_0, \vec{\boldsymbol{\lambda}}) \neq 0$ we can rule out the possibility that $\boldsymbol{\lambda}_0 = 0$.**
- **For our problem the optimal**

control is given by the rule

$$\tan\boldsymbol{\theta}^*(\boldsymbol{t}) = \boldsymbol{\kappa}(\boldsymbol{T} - \boldsymbol{t})$$

Note that $\tan\boldsymbol{\theta}^*(\boldsymbol{T}) = \mathbf{0}$.