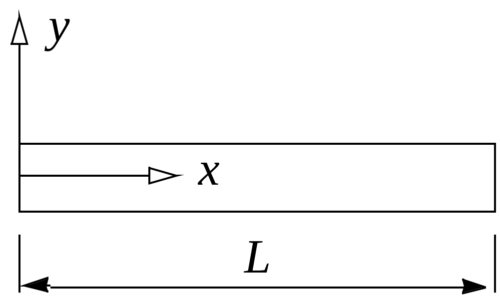
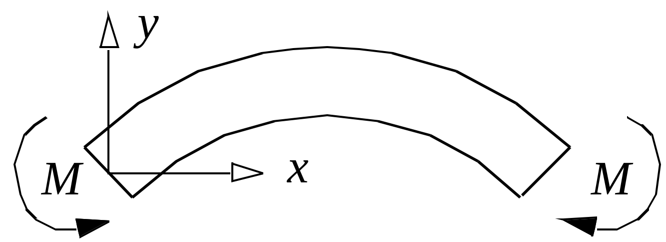
Lecture 5: Pure bending of beams

Each end of an initially straight, uniform beam is subjected to equal and opposite couples. The moment of the couples is denoted by M (units of F-L). It is shown in solid mechanics that the beam deforms into a plane circular arc.



undeformed beam



deformed beam

Neutral axis & cross sections remain plane

Some line elements originally parallel to the x-axis, loosely called fibers, are elongated and some are shortened in the deformed beam.

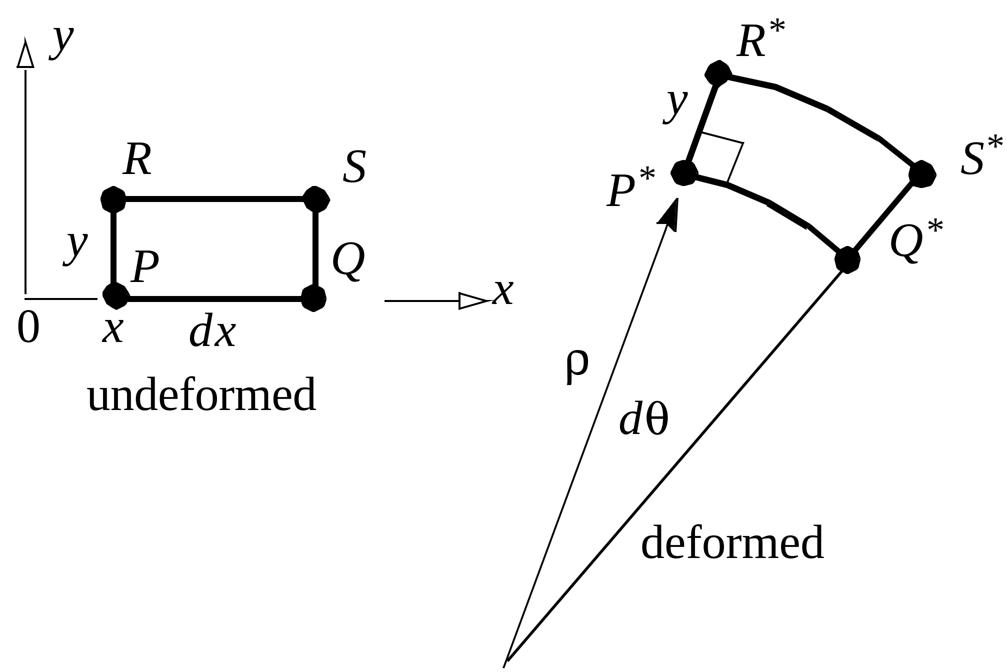
Hence, one fiber does not change length. It is called the neutral axis, and we take the z-axis in the undeformed beam to be the neutral axis.

The fiber on the neutral axis bends but does not change length.

Plane cross sections perpendicular to the x-axis in the undeformed beam remain plane and perpendicular to the x-axis in the deformed beam

Normal strain is linear in the y-coordinate

Show that: $$\varepsilon = \frac{y}{\rho}$$



ρ = radius of curvature of the neutral axis

$$\widehat{PQ} = \widehat{P^*Q^*}$$

$$\angle RPQ = \angle R^*P^*Q^* = \frac{\pi}{2}$$

$$\varepsilon = \frac{\widehat{R^*S^*} - \widehat{RS}}{\widehat{RS}}$$

$$\widehat{RS} = \widehat{PQ} = \widehat{P^*Q^*}$$

$$\varepsilon = \frac{\widehat{R^*S^*} - \widehat{P^*Q^*}}{\widehat{P^*Q^*}}$$

$$\varepsilon = \frac{(y+\rho)d\theta - \rho d\theta}{\rho d\theta}$$

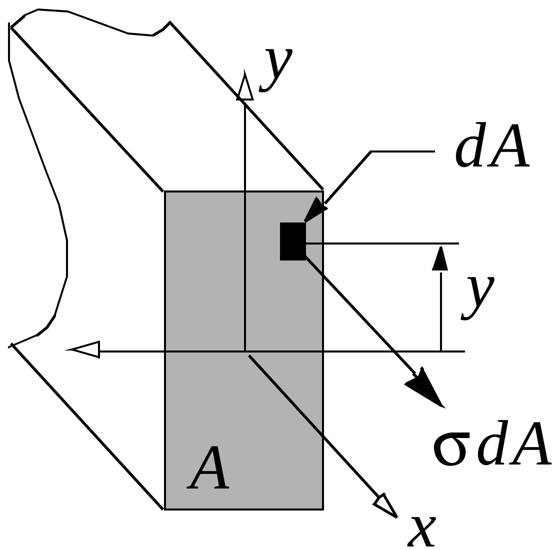
therefore: $$\varepsilon = \frac{y}{\rho}$$

Hooke's law

$$\sigma = E\varepsilon$$

Static equivalence

$$N = \iint_A \sigma \, dA \qquad M = \iint_A y\sigma \, dA$$



differential normal force

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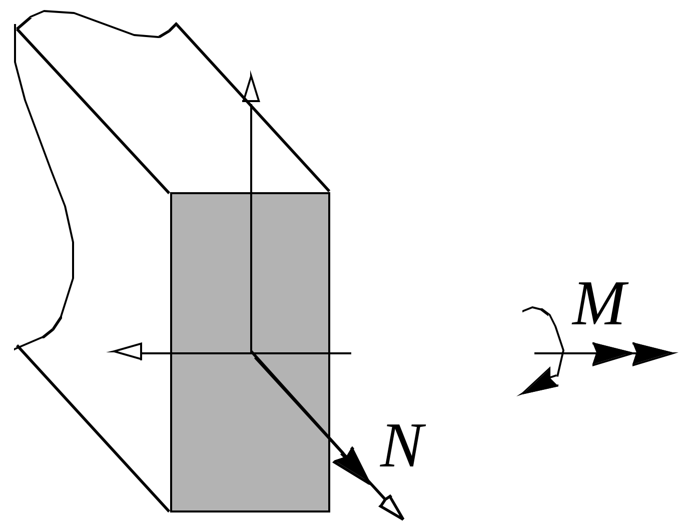


normal force and moment

Hooke's law (continued)

Substitute $\varepsilon = \frac{y}{\rho}$ and $\sigma = E\varepsilon$ into the *N* and *M* integrals, recognizing E and ρ are independent of integration over the cross section, to get

$$N = \frac{E}{\rho}\iint_A y\,dA \qquad M = \frac{E}{\rho}\iint_A y^2\,dA$$

But the net axial force $N = 0$ in pure bending.

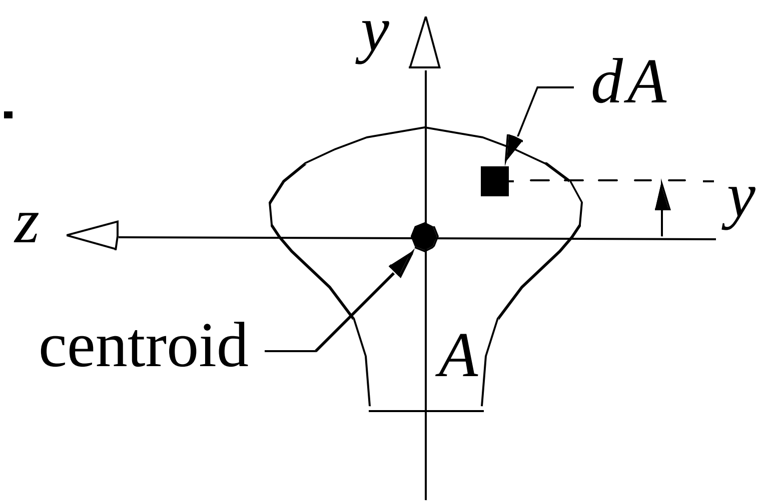
Therefore, we must have $\iint_A y\,dA = 0$; i.e., the origin of the cross-sectional coordinates is at the <u>centroid</u> of the area.

Hooke's law (continued)

The <u>second area moment</u> about z-axis through the centroidal coordinates in the cross section is denoted by I, where

$$I = \iint_A y^2 dA$$

The dimensional units of I are L^4; e.g.,in^4 or m^4.



Hooke's law (concluded)

Hence, for centroidal coordinates in the cross section we find

$$M = \frac{EI}{\rho} \qquad \text{or} \qquad \frac{1}{\rho} = \frac{M}{EI}$$

which is Hooke's law for the beam.

That is, Hooke's law for the beam relates the reciprocal of the radius of curvature of the x-axis in the deformed beam to the bending moment divided by the bending stiffness EI of the cross section.Beams with a large value of the bending stiffness have less curvature, or equivalently, a larger radius of curvature.

Flexure formula

Substitute Hooke's law $\frac{1}{\rho} = \frac{M}{EI}$ into $\sigma = E\varepsilon = E\frac{y}{\rho}$

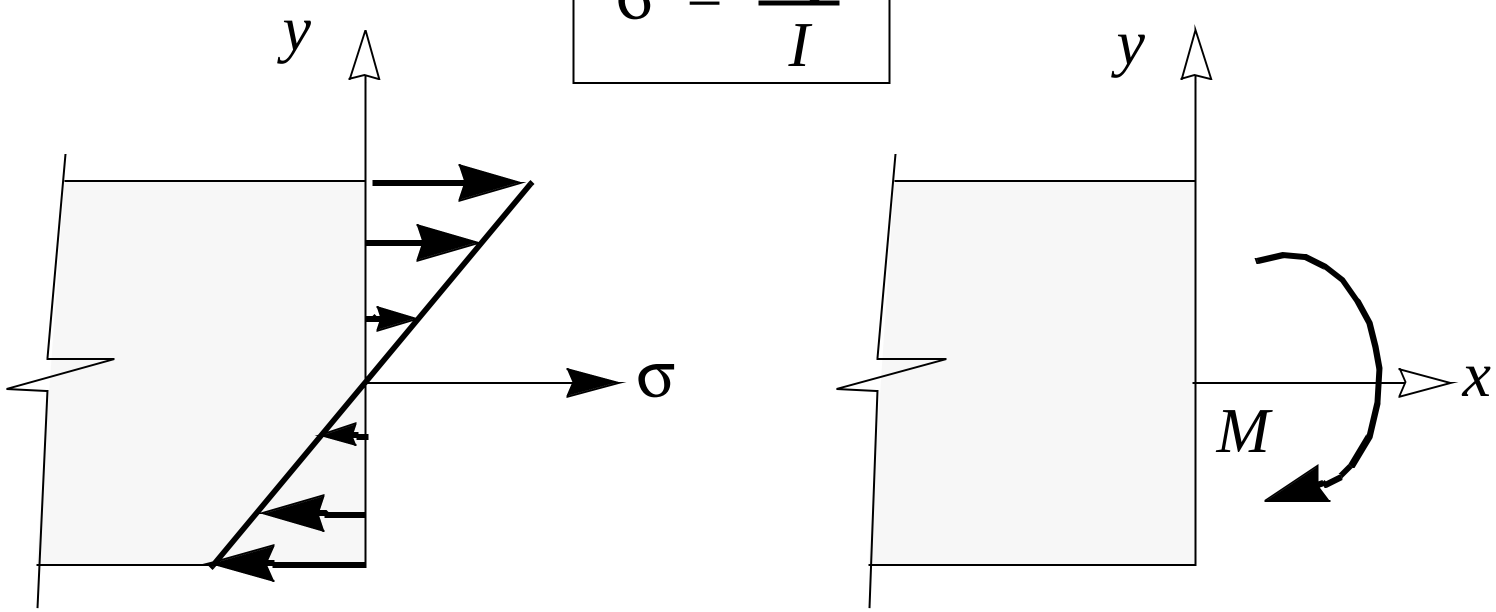
to get

$$\sigma = \frac{My}{I}$$

This is the flexure formula. It shows that the normal stress due to bending is proportional to the bending moment M, proportional to the y-coordinate in the cross section, and inversely proportional to the second area moment. At y = 0 the bending normal stress is zero, and in the section the max magnitude $\sigma_{max} = \frac{|M||y|_{max}}{I}$.

Bending normal stresses

$$\sigma = \frac{My}{I}$$



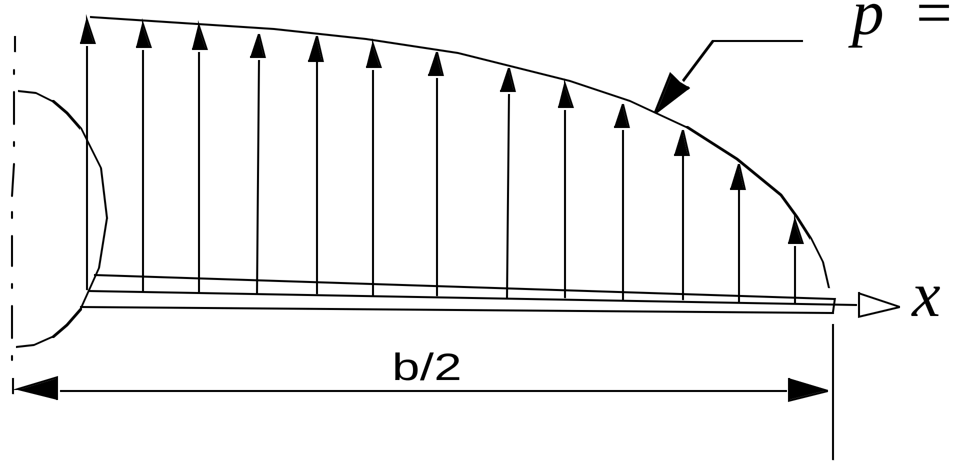
For M > 0 the upper fibers (y > 0) are in tension and the lower fibers (y < 0) are in compression. If M < 0, then the upper fibers are in compression and the lower fibers are in tension.

Bending is a combination of tension and compression.

Normal stresses in wing bending

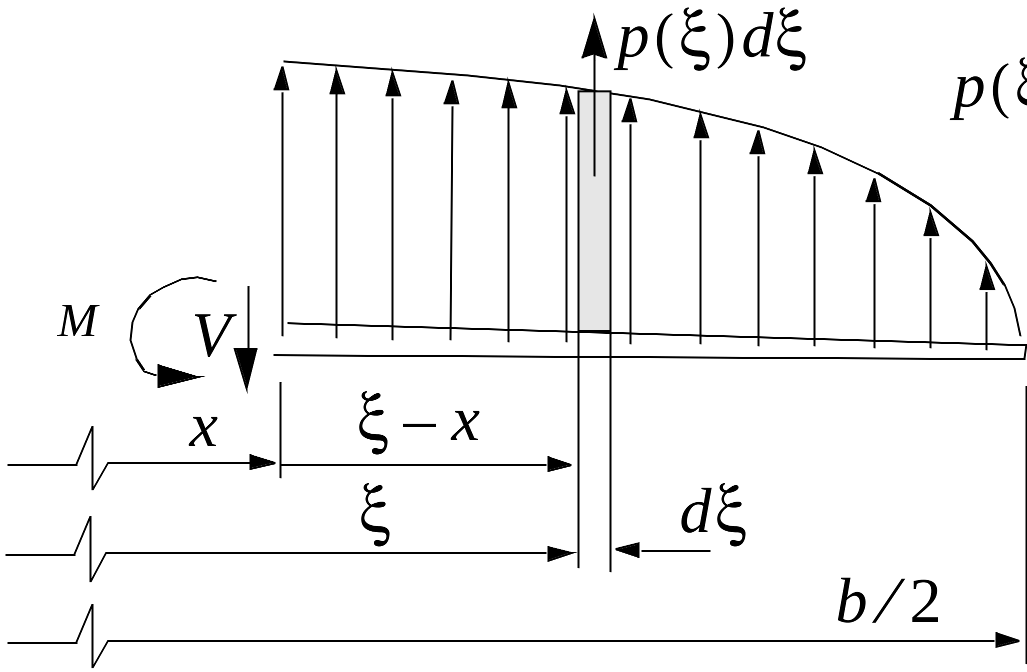
p = airload intensity in lb/ft

$$p = \frac{2L}{\pi\left(\frac{b}{2}\right)}\sqrt{1-\left(\frac{2x}{b}\right)^2}$$



Steady level flight with L = total lift (both wings), and b/2 the semi-span. How do we size the spar?

Shear force and bending moment



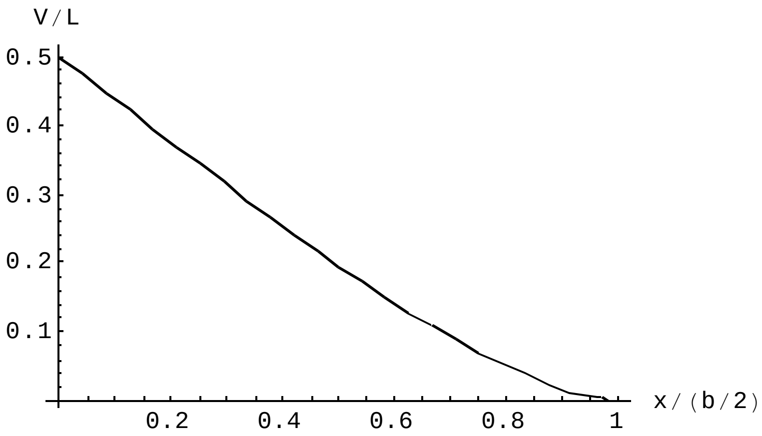
FBD & Equilibrium

$$p(\xi) = \frac{2L}{\pi(b/2)}\sqrt{1-\left(\frac{2\xi}{b}\right)^2}$$

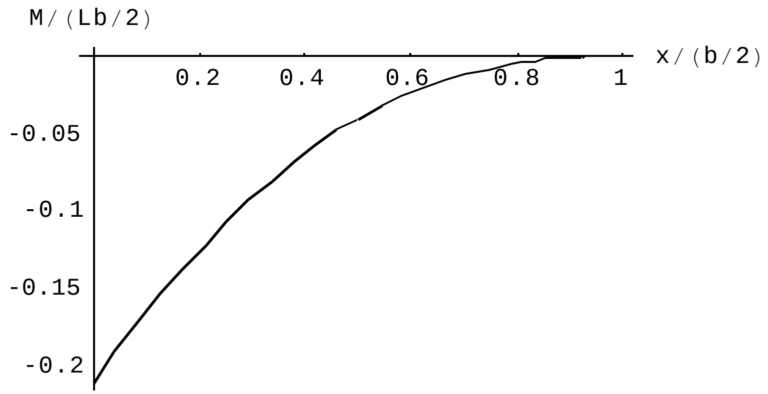
$$V(x) = \int_x^{b/2} p(\xi)d\xi$$

$$M(x) = -\int_x^{b/2} (\xi - x)p(\xi)d\xi$$

Shear force and bending moment distribution



shear force distribution



bending moment distribution

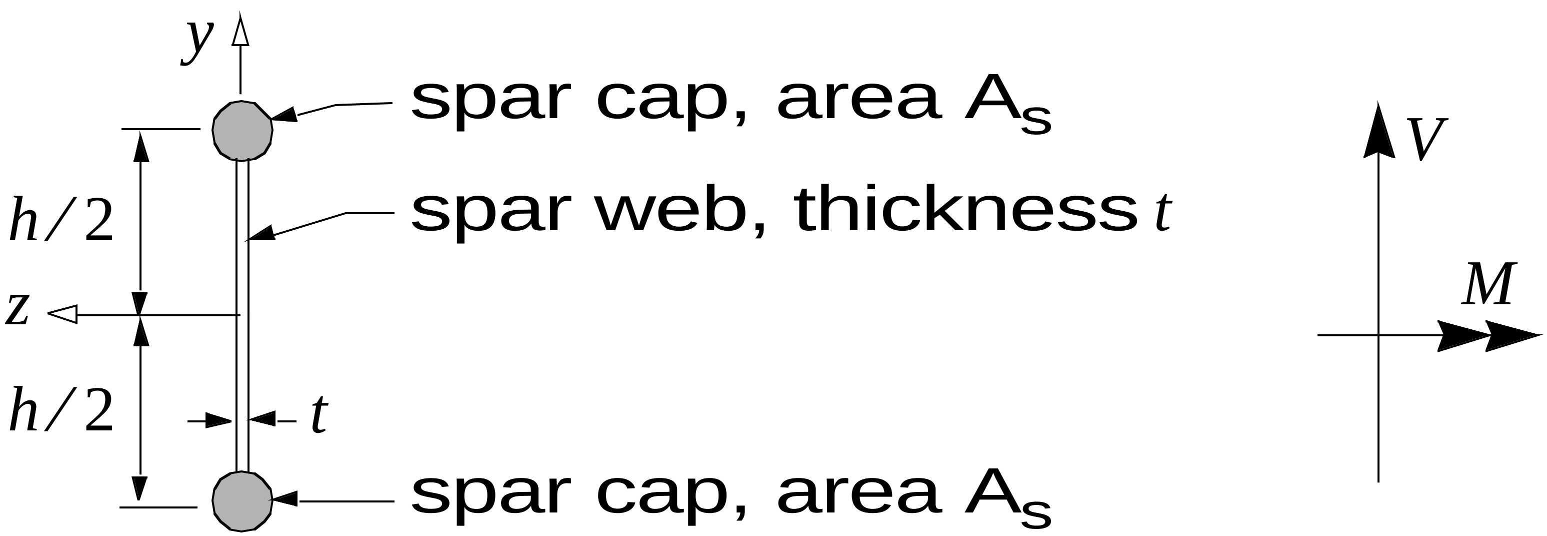
$$V(x=0) \; = \; L/2 \qquad\qquad M(x=0) \; = \; -(bL)/(3\pi)$$

For $L \; = \; 20,000$ lb and $b \; = \; 65$ ft, at the wing root

$$V \; = \; 10,000 \text{ lb and } M \; = \; -137,934 \text{ lb-ft}$$

Wing spar: thickness t and spar cap area A_s

The magnitude of the maximum shear force and bending moment are at the wing root. So we wish to size the spar to carry these loads. Idealized spar.



Usually dimension h is given by the thickness to chord ratio for the airfoil.

Spar sizing

Requirement

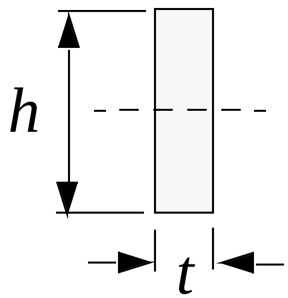
Select area A_s and thickness t such that magnitude of the normal stresses in the spar are less that the yield stress of material.

We need formulas for the normal stresses due to bending and the shear stresses due to bending.

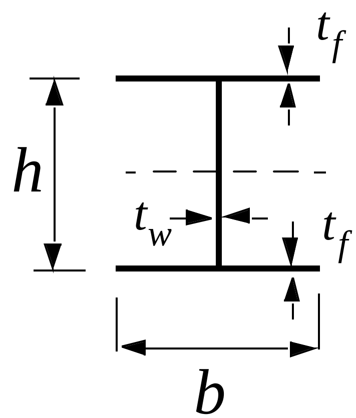
The formula for the normal stress due to bending is called the flexure formula, and we already derived this important formula as

$$\sigma = \frac{My}{I}$$

Second area moments for some sections

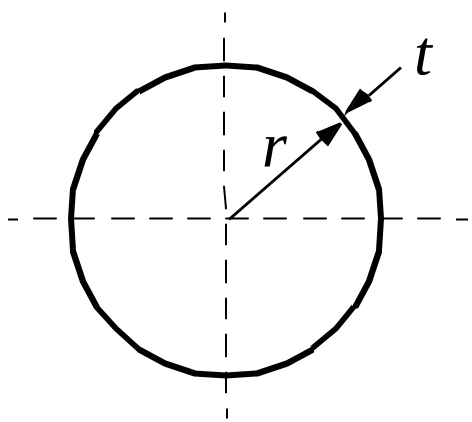


$$I = \frac{h^3 t}{12}$$



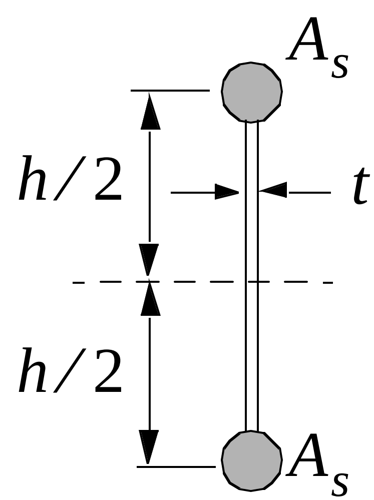
$$I = \frac{h^2 b t_f}{2} + \frac{h^3 t_w}{12}$$

thin-walled I-section



$$I = \pi r^3 t$$

thin-walled tube



$$I = \frac{h^2 A_s}{2} + \frac{h^3 t}{12}$$

spar

Spar cap areas

Say the thickness to chord ratio is 0.14 of the airfoil at the wing root with a chord of 8 ft. So h = 1.12 ft. for the spar. The material yield strength is 42 ksi (2024-T4), and assume a factor of safety of 1.5 against yielding in the steady level flight condition. Then the allowable stress is 28 ksi. In the flexure formula, the maximum bending normal stress occurs at y = ± h/2 at the wing root where M = -137,934 lb-ft. For a thin-walled web we neglect the contribution of the web to the second area moment and take $I = h^2 A_s/2$.

$$\sigma_{max} = \left|\frac{M(h/2)}{(h^2 A_s)/2}\right| = \frac{M}{hA_s}$$

Spar cap areas (concluded)

So

$$28000\text{lb/in}^2 = \frac{(137,934\text{ lb-ft})}{(1.12\text{ ft})A_s}$$, or

$$A_s = \frac{(137,934\text{ lb-ft})}{(1.12\text{ ft})\left(28000\dfrac{\text{lb}}{\text{in}^2}\right)}$$

$$A_s = 4.40\text{ in}^2$$