Introduction to Aerospace Engineering

5. Aircraft Performance

5.1 Equilibrium Flight

In order to discuss performance, stability, and control, we must first establish the concept of equilibrium flight. For now, lets consider a generic equilibrium climbing flight condition. For the following we will make one simplifying assumption, the thrust is aligned with the velocity vector. If the vehicle is in a climb, at a flight path angle γ (pitch angle of wind axes system), then the force equations along the wind x^w and z^w axes are given by:

$$x^w: \qquad T - D - W\sin\gamma = 0$$

$$-z^w: \qquad L - W\cos\gamma = 0$$

In addition, we have the moment balance:

$$M = 0$$

Of course the remaining forces and moments must be in balance so we have:

$$L_{(roll)} = 0 \qquad\qquad N_{(yaw)} = 0 \qquad\qquad Q_{(side)} = 0$$

For the ***SPECIAL CASE***, of ***straight and level flight***, ($\gamma = 0$) the equations of interest become:

$$T = D$$

$$L = W$$

$$M = 0$$

These are the equations that we use for determining performance characteristics (primarily the two equations dealing with lift and drag), and stability characteristics (primarily the pitch-moment and lift equations).

For now we will assume that we can set the thrust to a value that will balance the drag so that the T = D equation is satisfied. We will now look at the L = W equation. Recall that the lift coefficient is given by the expression:

$$C_L = \frac{L}{1/2\rho V^2 S}$$

Since L = W, we can note that the expression for the lift coefficient in straight and level equilibrium flight is given by:

$$C_L = \frac{W}{1/2\rho V^2 S}$$

Here we can make some interesting observations. In a vehicle, the weight is nearly constant so we an see that, at a given altitude (ρ = const), that as V increases, C_L must decrease and as V decreases, C_L must increase. So at higher speed we have a lower lift coefficient, and at lower speed we have a higher lift coefficient, the lift, however, ***stays the same*** (=W). We can rearrange the equation to determine the speed required as a function of the lift coefficient:

$$V = \sqrt{\frac{W}{1/2\,\rho S C_L}}$$

This equation just tells us the same thing as we noted previously, increase C_L, then speed must decrease, and decrease C_L, then speed increases. If there were a maximum value for C_L then we see that there is a minimum speed at which we could fly. This speed would be the ***aerodynamic*** lower limit and is called the stall speed or $V_{stall.}$

$$V_{\min_{aero}} = V_{stall} = \sqrt{\frac{W}{1/2\,\rho S C_{L_{\max}}}}$$

We now need to investigate how the lift coefficient can be changed. The lift coefficient changes with angle-of-attack. For low angles-of-attack it changes linearly and can therefor be represented by the equation of a straight line. Hence we can mathematically model the lift coefficient with the following equations:

$$C_L = C_{L_0} + \frac{\Delta C_L}{\Delta \alpha}\alpha$$

$$= \frac{\Delta C_L}{\Delta \alpha}\bar{\alpha}$$

where

$$\bar{\alpha} = \alpha - \alpha_{0L}$$

and

$$C_{L_0} = -\frac{\Delta C_L}{\Delta \alpha}\alpha_{0L}$$

We generally define the lift curve as the plot of C_L vs α, and the slope of that curve we designate as the "lift-curve slope." Hence we can define:

$$a = \frac{\Delta C_L}{\Delta \alpha} = \frac{dC_L}{d\alpha}$$ The lift-curve slope.

Typical values are always less than 2π / rad.

As an example we can consider an aircraft with the following characteristics:

W = 17578 lbs $S = 260\ ft^2$ b = 27.5 ft $\bar{c}$ = 10.8 ft.

Consider it to be flying level at 35,000 ft at Mach = 0.8. Find the lift coefficient and angle of attack if $\alpha_{0L} = -0.2°$ and the lift-curve slope is 4 / rad.

At 35,000 ft. the density is 0.0007382 slugs/ft^3 and the speed of sound is 973.14 ft/sec.

V = M_a (speed of sound) = 0.8 (973.14) = 778.52 ft/sec

$$C_L = \frac{L}{1/2\rho V^2 S} = \frac{W}{1/2\rho V^2 S} \qquad \text{(L = W in level flight)}$$

Then

$$C_L = \frac{17575}{1/2(0.0007382)(778.14^2)(260)} = 0.3023$$

The related angle-of-attack can be determined from the equation of the lift coefficient:

$$C_L = a\bar{\alpha} = 4(\bar{\alpha}) = 0.3023 \qquad \Rightarrow \qquad \bar{\alpha} = 0.07557 \text{ rad} = 4.33 \text{ deg}$$

$$\bar{\alpha} = \alpha - \alpha_{0L} = \alpha - (-2) = \alpha + 2 = 4.33 \qquad \Rightarrow \qquad \alpha = 2.33°$$

Hence the aircraft is flying at an angle-of-attack of 2.33 deg at Mach 0.8 at 35,000 ft. The angle-of- attack from the zero-lift condition is 4.33 deg.

5.2. **Performance**

Performance considerations are primarily concerned with the force equations. If we consider straight and level flight, we can sum up the expressions for equilibrium level flight:

Level Flight: T = D ; L = W

In order to study these equations in depth, we need to generate mathematical models for the thrust, lift, and drag. We already have the required model for lift and drag, so we will need to develop a model for thrust.

Thrust Models: The T in the above equation!

For engines that drive a propeller, the following model is reasonable as long as it is not used at very low speeds.

Propeller: (piston engines and turboprops)

$$T = SP_{SL}\frac{\rho}{\rho_{SL}}\frac{\eta_p}{V_\infty}$$

where:

T = thrust (lbs, N)
SP = shaft power (ft lb/sec, watts)
ρ = density
η_p = propeller efficiency
V_∞ = Speed of aircraft (free stream velocity)
$(\bullet)_{SL}$ = sea level value

Example: What is the thrust of a Cessna Cardinal RG if it has a 200 horsepower engine and is flying at 10,000 feet altitude at 142.195 knots?

142.195 knots * 1. 6878 ft/sec/knot = 240 ft/sec

$$T = SP_{SL} \frac{\rho}{\rho_{SL}} \frac{\eta_p}{V_\infty}$$

$$T = 200\text{hp} * 550\text{ft lb/sec hp} * \frac{0.001756}{0.002377} * \frac{0.9}{240\text{ft/sec}}$$

$= 304.7$ lbs

If the Cardinal weighs 2800 lbs, and assuming level flight, what is the lift to drag ratio (L/D) ?

T = D = 304.7 lbs

$$L/D = \frac{2800}{304.7} = 9.2 \qquad L = W = 2800 \text{ lbs}$$

Thrust model, Turbojet - Turbofan

Basic Model:

$$T = T_{SL} \frac{\rho}{\rho_{SL}}$$

with afterburner:

$$T = T_{SL} (1 + 0.7 M_\infty) \frac{\rho}{\rho_{SL}}$$

where M_∞ = free stream Mach number (aircraft Mach number)

Example: At what altitude will a turbojet be at half its rated thrust level?

From altitude tables: ρ/ρ_{SL} = 0.5 at ~ 22,000 ft.

$\rho = \rho_{SL}$ *0.5 = 0.002377* 0.5 = 0.001188 slug/ft^3

ρ at 22,000 ft = 0.001184 slug/ft^3

A bound on minimum airspeed (not necessarily ***the*** bound)

We have already discussed a lower bound on the airspeed. It is useful to review it here. If we consider level flight, we have the well-known requirements:

Level Flight - T = D;   L = W!

Hence: $$L = W = C_L \frac{1}{2} \rho V^2 S$$

where:

- C_L = lift coefficient
- ρ = local density at altitude
- V = aircraft speed (relative wind)
- S = Reference area (planform area of wing)

NOTE: IN THIS EQUATION *W* IS CONSTANT (FIXED VALUE)

Therefore, ***at a given altitude*** (level flight) if V goes down, C_L must go up ! (and vice versa)

A ***possible limiting*** factor on minimum speed is $C_{L_{\max}}$

$$V_{stall} = \sqrt{\frac{W}{\frac{1}{2} \rho C_{L_{\max}} S}}$$

Now lets consider the drag equation. Again for level flight we require:

Level Flight T = D;   L = W

Now we are interested in how drag behaves as the velocity changes. That is, we want to develop a mathematical model for drag. We will find that drag behaves in an unexpected way!,

Drag - Its *STRANGE* behavior!

The equation for drag can be written in terms of its drag coefficient as:

$$D = C_D \, \frac{1}{2} \rho V^2 S.$$

However, it has been mentioned previously that we don't get something for nothing. We pay the price for supporting the aircraft with lift. The price we pay is what is called induced drag, a part of the drag that is caused by the lift. From experiments, we can observe that the drag coefficient varies parabolically (in most cases) with the lift coefficient. Consequently we can model the drag coefficient in the following way:

$$C_D = C_{D_{0L}} + K C_L^2$$

So, the drag (coefficient) has a term that depends on Lift (coefficient)! Automobiles depend on the ground to support them hence they only have the first term, and automobile drag (force) goes up with V^2. The first term is called the parasite drag coefficient or loosely the "zero lift" drag coefficient, and the second term is called the "induced drag" coefficient. It contains the parameter K that depends primarily on the geometry of the aircraft wing (since that provides most of the lift)

In this model, we can define the following ingredients of the drag coefficient:

$C_{D_{0L}}$ = "zero lift" drag coefficient

$K = \dfrac{1}{\pi AR e_0}$ "Induced drag parameter"

where

AR = wing aspect ratio = $(\text{span})^2/S$

e_0 = Oswald efficiency factor (a measure of the efficiency of the A/C

We can now substitute these values into the drag equation and carry through the algebra: Then

$$D = \left(C_{D_{0L}} + K C_L^2\right) \frac{1}{2} \rho V^2 S$$

or

$$D = C_{D_{0L}} \frac{1}{2} \rho V^2 S + K C_L^2 \frac{1}{2} \rho V^2 S$$

but

$$C_L = \frac{W}{\frac{1}{2} \rho V^2 S}$$

so

$$\boxed{\begin{aligned} D &= C_{D_{0L}} \frac{1}{2} \rho V^2 S + \frac{K W^2}{\frac{1}{2} \rho V^2 S} \\ D &= C_{D_{0L}} \bar{q} S + \frac{K W^2}{\bar{q} S} \end{aligned}}$$

where $\bar{q} = \frac{1}{2} \rho V^2$

In order to calculate the drag, we need certain information specific to the aircraft of interest. In particular, we need the zero-lift drag coefficient, $C_{D_{0L}}$, and the induced drag parameter, K. This information can be given in several different ways. One of the most convenient is in terms of a ***Drag Polar***. A drag polar is defined as an expression for drag in terms of the lift coefficient. Typically, for first approximations, we assume a parabolic drag polar as described in the previous section. The drag polar then has the form:

Drag Polar:

$$C_D = C_{D_{0L}} + K C_L^2$$

For our example let us assume that the following values are given:

$C_{D_{0L}} = 0.02$ K = 0.04

Then the drag polar looks like:

$$C_D = 0.02 + 0.04\, C_L^2$$

In order to do any calculations, additional information is required regarding the geometry of the vehicle of interest. For this problem assume that we are dealing with a aircraft with the following characteristics:

$$W = 25{,}000 \text{ lbs}$$
$$S = 375 \text{ ft}^2$$

In addition we need density, ρ, at altitude:

$\rho_{SL} = 0.002377$ slugs/ft^3
$\rho_{10} = 0.001756$
$\rho_{20} = 0.001267$
$\rho_{30} = 0.000891$
$\rho_{40} = 0.000587$
$\rho_{50} = 0.000364$
$\rho_{60} = 0.000226$

The final bit of information that we need is that which describes the thrust behavior with airspeed. For our purpose we will assume the simplest model of a turbofan engine.

Thrust Model: (*special case*)

Assume - Low by-pass turbofan engine, whose thrust is independent of speed. It will also be assumed that the thrust varies proportional to density. Under these circumstances, the model for the engine thrust is given by:

$$T = T_{SL} * \left(\frac{\rho}{\rho_{SL}} \right)^x,$$

where x = 1.

In addition, we can introduce a throttle setting that allows us to operate at any thrust level we desire. A simple way to do that is to assume the throttle can be set to give us some fraction of the total thrust. Hence we can model the throttle contribution to thrust with the simple equation:

$$T_{SL} = T_{max} * \delta_T \qquad 0 \le \delta_T \le 1,$$

where

δ_T = Thrust level (throttle setting)

Hence ***THRUST AVAILABLE*** is modeled by:

$$T = T_{max}\,\frac{\rho}{\rho_{SL}}\,\delta_T \qquad = \text{ constant for given altitude}$$

Plot:

Thrust available vs. Airspeed
At different altitudes for given throttle setting

Example Problem: Determine the flight envelope of an aircraft with the following characteristics:

Drag Polar:

$$C_D = 0.02 + 0.04\,C_L^2$$

Maximum Lift Coefficient:

$$C_{L_{\max}} = 1.5$$

Weight, W = 25,000 lbs
Wing Area, S = 375 ft^2

Hence:

$C_{D_{0L}}$ = 0.02
K = 0.04

Thrust Properties: Low-pass Turbojet with full throttle thrust at sea level given by:

T_{SL} = 4000lbs = const (Independent of speed)

Solution: Find the maximum and minimum flight speeds at various altitudes:

1. Calculate and plot ***Drag vs Velocity*** at different altitudes. (Thrust required)

2. Calculate and plot ***Thrust vs Velocity*** at the same altitudes as in (1). (Thrust available)

3. Calculate ***Stall speed vs. Altitude***

4. From T = D determine the Thrust-limited velocities for both the maximum and minimum airspeeds (The points where the thrust and drag curves intersect)

5. The minimum speed is the greater of stall speed and the thrust-limited (T = D) minimum speed.

1. Calculate ***Drag (Thrust Required)***

$$D = C_{D_{0L}} \frac{1}{2} \rho V^2 S + \frac{K W^2}{\frac{1}{2} \rho V^2 S}$$

$$D = (0.02) \frac{1}{2} (0.001756)\, 500^2\, (375) + \frac{(0.04)\, 25000^2}{\frac{1}{2} (0.001756)\, 500^2\, (375)}$$

D = 1646.25 + 303.72 = 1949.97 lbs

Repeat for each altitude of interest and for range of airspeeds.

V = 100 ft/sec - 800 ft/sec (Values depend on aircraft)

Altitude	Density
0 (sea level)	0.002377 slug/ft^3
10,000 ft	0.001756
20,000 ft	0.001267
30,000 ft	0.000891
40,000 ft	0.000587

2. Calculate ***Thrust Available*** at same altitude and airspeed range

For low bypass turbojet, the thrust math model is given by:

$$T = T_{SL} \frac{\rho}{\rho_{SL}} \delta_T \qquad 0 \le \delta_T \le 1$$

For 10,000 ft maximum throttle setting,

$$T = 4000 \frac{0.001756}{0.002377} = 2954.98 \text{ lbs}$$

3. Calculate and plot ***Altitude vs Stall*** speed (L = W)

$$V_{stall} = \sqrt{\frac{W}{\frac{1}{2}\rho\, S\, C_{L_{max}}}}$$

$$V_{stall} = \sqrt{\frac{25000}{\frac{1}{2}0.001756\,(375)\,1.5}} = 225 \text{ ft/sec}$$

4. Pick off the plots where T = D and plot minimum and maximum speeds vs. altitude, with altitude as the ordinate (y axis).

5. Add stall velocity limits to graph, pick the envelope to be the greater of stall or thrust limited minimum speeds, and the maximum speeds.

In all these plots the value if the minimum drag at every altitude ***is the same***. We can calculate this value, the lift coefficient at which it occurs, the drag coefficient when it occurs, the L/D ratio at the min-drag point (since weight is a constant, it is L/D_{max}.), and corresponding airspeed at which it occurs (different for each altitude).

Calculating Minimum Drag

We must find the minimum drag, subject to the constraint that lift = weight. In this case the minimum drag would occur when L (= W) /D is a maximum, or D/L is a minimum.

$$\frac{D}{L} = \frac{C_D \frac{1}{2}\rho\, V^2\, S}{C_L \frac{1}{2}\rho\, V^2\, S} = \frac{C_D}{C_L}$$

But

$$\frac{C_D}{C_L} = \frac{C_{D_{0L}} + K\, C_L^2}{C_L}$$

where

$C_{D_{0L}}$ = Zero-lift drag coefficient (parasite drag)

K = 1/ π AR e_0 Induced drag parameter

Problem: Find the lift coefficient that minimizes D/L

$$\min_{C_L} \quad \frac{C_D(C_L)}{C_L} = \frac{C_{D_{0L}} + K C_L^2}{C_L}$$

$$\frac{d(C_D/C_L)}{dC_L} = \frac{C_L\,(2 K C_L) - (C_{D_{0L}} + K C_L^2)}{C_L^2} = 0$$

or

$$K C_L^2 - C_{D_{0L}} = 0$$

which gives:

$$C_{L_{md}} = \sqrt{\frac{C_{D_{0L}}}{K}} \qquad \text{Min-Drag Lift coefficient}$$

For our class example:

$C_{D_{0L}} = 0.02$, K = 0.04

min-drag lift coefficient = $C_{L_{md}} = \sqrt{\frac{0.02}{0.04}} = 0.707$

$$V_{md} = \sqrt{\frac{W}{\frac{1}{2}\rho\, C_{L_{md}} S}} = \sqrt{\frac{25000}{\frac{1}{2}(0.001756)\, 0.707\, (375)}} = 327.7 \text{ ft/sec at } 10{,}000\text{ft}.$$

Note that the corresponding minimum drag -drag coefficient is

$$C_{D_{md}} = C_{D_{0L}} + K C_{L_{md}}^2 = C_{D_{0L}} + K \left(\frac{C_{D_{0L}}}{K} \right)$$

$$= 2\, C_{D_{0L}} = 0.04$$

SUMMARY:

$$C_{L_{md}} = \sqrt{\frac{C_{D_{0L}}}{K}}, \qquad C_{D_{md}} = 2\, C_{D_{0L}}, \qquad \frac{L}{D}\Big|_{\max} = \frac{C_{L_{md}}}{C_{D_{md}}} = \frac{1}{2\sqrt{C_{D_{0L}} K}}$$

for example problem:

$$L/D|_{\max} = \frac{1}{2\sqrt{0.02\,(0.04)}} = 17.7$$