8.0 Fundamental Ideas for Space Considerations

In order to get to space, we need to expend energy. In particular we must raise the vehicle to the height of interest and, if we want to stay at that height (go into orbit) then we have to achieve a certain speed. Hence we have to expend energy to increase the potential energy plus increase the kinetic energy. The total energy at the end of the boost equals the energy at the beginning of the boost plus the energy expended, minus the energy lost (due to drag, etc). All this energy must come from the rocket motor. So let us look at the potential and kinetic energy of an object located as some altitude, h, above the Earth' surface, that has some speed, V.

Kinetic Energy

From physics, we find that the kinetic energy (the energy due to motion) is expressed as:

$$KE = \frac{1}{2} m V^2 \tag{1}$$

where m = mass of vehicle
V = velocity of vehicle

Often times when considering space vehicles we are interested in the specific kinetic energy or the kinetic energy per unit mass. We designate that with a T, or

$$T = \frac{1}{2} V^2 \tag{2}$$

Potential Energy

Since we are discussing the potential energy of a vehicle far from the Earth's surface, we must include the fact that gravity is not constant. In order to start this discussion, we will introduce Newton's law of Gravity. It states that the force between two bodies with mass is proportional to the product of the masses and inversely proportional to the square of the distance between them:

$$F = G \frac{m_1 m_2}{r^2} \tag{3}$$

where m_1 and m_2 = mass of the two bodies
r = distance between the center of mass of the two masses
G = universal gravitational constant

This force is called an inverse square gravitational force.

If we examine the force at the surface of the Earth where $r = R_e$, and we consider the Earth to be m_1 and the object of interest to be m_2 , we can write the weight of the object (at the Earth's surface) as

$$W_2 = m_2 g_0 = G \frac{m_1 m_2}{r_e^2} \tag{4}$$

or

$$g_0 = \frac{G m_1}{R_e^2} = \frac{\mu}{R_e^2} \qquad \text{and} \qquad g = \frac{\mu}{r^2} \tag{5}$$

where μ = $G m_1$ = gravitational parameter of the Earth
= 3.986004415×10^5 km^3/s^2 = 1.407644×10^{16} ft^3/sec^2

R_e = radius of Earth = 6378.1363 km = 2.09256×10^{13} ft
= 3963.1902 miles = 3443.9181 Nautical miles

It turns out that the value of μ is known more accurately than the parameter *G*, and is used more often. Then the surface value of gravity can be expressed in terms of μ or vice-versa:

$$g_0 = \frac{\mu}{R_e^2} \qquad \text{or} \qquad \mu = g_0 R_e^2 \tag{6}$$

We can see how the force varies with gravity, or if we consider the force per unit mass (or g), and use Eqs. (5 & 6) we have:

$$g = \left(\frac{R_e^2}{(R_e + h)^2} \right) g_0 \tag{7}$$

where:
g = acceleration due to gravity at any altitude, *h*
h = height above the Earth's surface
R_e = radius of the Earth
g_0 = acceleration due to gravity at the Earth's surface

Equation (7) tells us how the acceleration due to gravity varies with altitude above the Earth's surface. One might expect that the potential energy has a different form than that for constant gravity (for a constant gravity field, PE = mgh or per unit mass, PE = gh, where h is measured from ***some arbitrary reference datum***), and that is indeed true.

Potential Energy

The potential energy for an inverse square gravitational force is ***referenced to a datum plane at infinity***! Hence all potential energies are negative! Without proof, it can be shown that the potential energy is given (per unit mass)

$$PE/m = U = -\frac{\mu}{r} \tag{8}$$

Consequently, the total mechanical energy of a spacecraft or rocket is given by:

Energy equation (per unit mass)

$$En = (KE + PE)/m = T + U = \frac{1}{2}V^2 - \frac{\mu}{r} \tag{9}$$

where V = velocity of vehicle
μ = gravitational parameter of the central body (Earth, or Sun, planet)
r = distance from center of central body.

Consequently, the rocket engine must provide the energy to change the total energy from that at the start of the mission (rocket vehicle on ground) to that at the end of the mission (rocket vehicle in orbit for example), and also overcome some losses due to aerodynamic drag. For now let us just consider the rocket contribution to changing the kinetic energy (***that is, lets neglect the gravity and aerodynamic forces***). If we do that, then we can write Newton's second law as:

$$T = m\frac{dV}{dt} \tag{10}$$

where m = mass of vehicle
T = rocket thrust (the only force considered, remember gravity and aero forces are neglected)
V = rocket speed
t = time
$\frac{dV}{dt}$ = rocket acceleration

Now it is known that the rocket thrust is related to the mass flow rate and the exit velocity of the gasses. The resulting thrust is expressed by the relation:

$$T = -\dot{m}V_e + (P_e - P_a)A_e \tag{11}$$

where V_e = exit gas velocity ***<u>relative to the rocket</u>***
$\dot{m}$ = the rate of change of the ***mass of the rocket*** due to the gasses leaving it. Note that it is a negative quantity because the rocket mass is decreasing!

P_e = nozzle exit pressure

P_a = ambient (atmospheric) pressure

A_e = nozzle exit area

The engine is most efficient when the contribution from the pressure terms is zero. We will assume that such is the case and if not, close enough to neglect the pressure terms compared to the so-called momentum thrust terms. If we substitute Eq. (11) (without the pressure terms) into Eq. (10) we have:

$$T = -\dot{m}V_e = -\frac{dm}{dt}V_e = m\frac{dV}{dt} \tag{12}$$

After rearranging we have:

$$dV = -V_e\frac{dm}{m} \tag{13}$$

And finally, we can integrate between the initial mass, m_1 and the final mass, m_2: and obtain the so-called ***rocket equation***

Rocket Equation (no gravity or aerodynamic forces)

$$V_2 - V_1 = \Delta V = V_e \ln\frac{m_1}{m_2} \tag{14}$$

The quantity $\frac{m_1}{m_2}$ is called the ***mass ratio*** (one of many different mass ratios). Note that if the mass ratio is greater than e = 2.7183, then the final velocity will be more than the exit velocity of the gasses, which is usually the case. We now need to address the issue of the exit velocity.

Specific Impulse

The specific impulse can be defined as the thrust divided by the ***weight*** flow rate, or

$$I_{sp} = \frac{T}{\dot{W}_p} \tag{15}$$

where I_{sp} = specific impulse, (has units of seconds)

$\dot{W}_p$ = propellent weight flow rate

However, the important thing for us to know is that the exit velocity of a rocket is directly related to the specific impulse of a given fuel. The relation is quite simple:

$$V_e = g_0 I_{sp} \tag{16}$$

Consequently, if we know the specific impulse for a given propellant combination, we can determine the exit velocity of the exhaust, and use Eq. (14) to determine the speed after a given amount of propellant is burned. Some values for typical liquid fuels are:

Propellant	I_{sp} (seconds)
liquid oxygen, kerosene (LOX/RP-1)	310
liquid oxygen, liquid hydrogen (LOX/LH_2)	455
fluorine, hydrogen (F_2/H_2)	465
nitrogen tetroxide, unsymmetrical dimethylhydrazine (N_2O_4/UDMH) (storable fuel)	290
typical solid fuels	170 - 250
plasma jet, arc jet	300 - 700

If we go to ion propulsion, the I_{sp} can be 10,000! However, the thrust level is at a small fraction of a Newton, so that fuel cannot be used on launch vehicles, but it works quite well in space.

Mass ratios

If we consider a rocket, we can define many different mass ratios. To start, we can divide the rocket mass into several different types:

$$m_1 = m_p + m_s + m_f \tag{17}$$

where m_1 = initial mass of rocket
m_p = mass of payload
m_s = mass of structure
m_f = mass of fuel

It is clear that the final mass is just the structural mass plus the payload mass:

$$m_2 = m_s + m_p = m_1 - m_p \tag{18}$$

The ratio of the payload mass over the total initial rocket mass is called the ***payload ratio***. Define the ***payload ratio***, or ***payload mass fraction,*** λ_p:

$$\lambda_p \equiv \frac{m_p}{m_1} \tag{19}$$

and the ***step or effective structural factor*** or ***structural ratio***, λ_s:

$$\lambda_s = \frac{m_s}{m_s + m_p} \tag{20}$$

Note that the structural ratio does not include the payload in the denominator. In this way, the structural ratio is associated with the vehicle and does not have to be changed every time the payload is changed. However, sometimes the ***overall structural mass fraction***, $\bar{\lambda}_s$, is defined as:

$$\bar{\lambda}_s = \frac{m_s}{m_1} \tag{21}$$

Then the relation between the effective structural factor and the overall structural mass fraction is determined from:

$$\lambda_s = \frac{m_s}{m_s + m_f} = \frac{m_s}{m_1 - m_p} = \frac{\bar{\lambda}_s}{1 - \lambda_p} \qquad \text{or} \qquad \bar{\lambda}_s = (1 - \lambda_p)\,\lambda_s \tag{22}$$

We can define the ***total fuel (total propellant) mass fraction,*** λ_f**:**

$$\lambda_f = \frac{m_f}{m_1} \tag{23}$$

It is clear that the following relation is true:

$$\frac{m_s}{m_1} + \frac{m_f}{m_1} + \frac{m_p}{m_1} = \bar{\lambda}_s + \lambda_f + \lambda_p = 1 \tag{24}$$

The mass ratio of the initial mass over the final mass, $\frac{m_1}{m_2}$, was defined previously, but we can now write it in several ways:

Mass Fraction

$$
\begin{aligned}
\frac{m_1}{m_2} &= \frac{m_p + m_s + m_f}{m_p + m_s} = \frac{m_1}{m_1 - m_f} = \frac{1}{1 - \lambda_f} \\
&= \frac{1}{1 - \frac{m_f}{m_1} \cdot \frac{(m_s + m_f)}{(m_s + m_f)}} = \frac{1}{1 - \left(\frac{m_s + m_f}{m_1}\right)\left(\frac{m_f}{m_s + m_f}\right)} \\
&= \frac{1}{1 - (1 - \lambda_p)(1 - \lambda_s)} \\
&= \frac{1}{\lambda_s + (1 - \lambda_s)\lambda_p}
\end{aligned}
\qquad (25)
$$

The mass ratio takes on its largest value when there is ***no payload*** ($\lambda_p = 0$). Then the maximum increase or upper bound for the change in velocity for a rocket is:

$$
\Delta V_{\mathrm{max}} = V_e \ln \frac{1}{\lambda_s} \qquad (26)
$$

If we assume that current technology provides a structural ratio of about $\lambda_s = 0.10$ then the maximum speed obtained from rest would be about 2.3 times the exhaust velocity.

The payload mass fraction can be written in many ways as a function of other mass fractions. Consider:

$$
\lambda_p = \frac{m_p}{m_1} = \left(1 - \bar{\lambda}_s\right) - \lambda_f = 1 - (1 - \lambda_p)\lambda_s - \lambda_f
$$

and solve for λ_p to get:

$$
\lambda_p = \frac{1 - \lambda_s - \lambda_f}{1 - \lambda_s} = 1 - \frac{\lambda_f}{1 - \lambda_s} \qquad (27)
$$

One of the major design objectives for spacecraft and boosters is to obtain the highest possible payload ratio or equivalently, given the gross lift-off weight (mass) maximize the payload weight (mass), or given the payload weight (mass), minimize the gross lift-off weight (mass).

We can ***estimate*** the time to burn by making a few assumptions and using what we know. Consider, for example, the thrust to weight ratio of a vehicle:

$$\frac{T}{W_1} = \frac{|\dot{m}|\,V_e}{W_1} = \frac{|\dot{m}|\,g_0 I_{sp}}{m_1\,g_0}$$

now make the assumption that $|\dot{m}| \approx \frac{m_f}{t_b}$ and substitute it into the above equation to get after rearranging:

$$t_b = \frac{\lambda_f I_{sp}}{T/W_1} \qquad (28)$$

Example: Consider a spacecraft with a gross weight of 6896 N (1552 lbs) (mass of 703 kg) that is to be the payload of a single stage booster capable of a ΔV= 7930 m/s. The I_{sp}= 350 s. (From Hale, *Introduction to Space Flight*, Prentice Hall, 1994)

a) Estimate the Gross lift-off weight

Lets first calculate the things we know and see where it gets us. We can get an estimate of the mass ratio from the "rocket equation" if we neglect the gravity and atmosphere effects. In order to use that equation, we need to have the exit or exhaust velocity:

$$V_e = g_0\, i_{sp} = 9.807\,(350) = 3432.45 \text{ m/s}$$

Then the mass ratio is obtained from the rocket equation:

$$\Delta V = Ve \ln\frac{m_1}{m_2} = 3432.45 \ln\frac{m_2}{m_2} = 7930 \qquad \Rightarrow \qquad \frac{m_1}{m_2} = 10.077$$

Now the final mass include both the payload and the structural mass, so we must select a structural mass ratio before we can determine the lift-off mass. Or, we can look at a range of structural mass ratios and see how the initial gross lift-off weight (GLOW) is affected by the structural ratio (or step structural ratio). We can estimate the fuel ratio from the mass ratio as follows:

$$\frac{m_1}{m_2} = \frac{m_1}{m_1 - m_f} = \frac{1}{1 - m_f} \qquad \Rightarrow \qquad \frac{m_f}{m_1} = \lambda_f = \frac{m_1/m_2 - 1}{m_1/m_2} = \frac{10.077 - 1}{10.077} = 0.901$$

Hence the propellant will take 90% of the booster initial mass (weight). We can now estimate the payload fraction, but we note that given the fuel mass ratio, it depends only on the structural ratio. From Eq. (27) we have:

$$\frac{m_p}{m_1} = \lambda_p = 1 - \frac{\lambda_f}{1 - \lambda_s} = 1 - \frac{0.901}{1 - \lambda_s}$$

or rearranging:

$$m_1 = \left(\frac{1 - \lambda_s}{0.099 - \lambda_s} \right) m_p$$

We can see that the conventional state-of-the-art structural ratio of $\lambda_s = 0.10$ will not work here so we will have to go to a technology-pushing value of $\lambda = 0.095$. With this value, we have

$$m_1 = \left(\frac{1 - 0.095}{0.099 - 0.095} \right) 703 = 226.6\,(703) = 159{,}054 \text{ kg} \Rightarrow 1{,}559{,}842 \text{ N} = 350{,}668 \text{ lbs}$$

The fuel mass is

$$m_f = \lambda_f m_1 = 0.901\,(159054) = 143{,}308 \text{ kg} \Rightarrow 1{,}405{,}418 \text{ N} = 315{,}952 \text{ lbs}$$

and the structural mass is

$$m_s = m_1 - m_f - m_p = 159{,}054 - 142{,}308 - 703 = 15{,}043 \text{ kg} \Rightarrow 147{,}527 \text{ N} = 33{,}165 \text{ lbs}$$

If we use some unobtainium for our structure, so that we can reduce the structure fraction to 0.05, then the initial mass becomes:

$$m_1 = \left(\frac{1 - 0.05}{0.099 - 0.05} \right) 703 = 19.388\,(703) = 13{,}630 \text{ kg} \Rightarrow 133{,}665 \text{ N} = 30{,}050 \text{ lbs}$$

and we reduce the initial mass by an order of magnitude!

b) If the initial launch thrust to weight ratio was 1.5, what would be the lift-off thrust, and what would be the approximate burn time.

$$T_1 = \frac{T}{W_1} W_1 = 1.5\,(1559842) = 2{,}339{,}763 \text{ N} = 562{,}000 lbs$$

The estimated time to burn is:

$$t_b = \frac{\lambda_p I_{sp}}{T/W_1} = \frac{0.901\,(350)}{1.5} = 210 \text{ s}$$

Multistage Rockets

It can be seen from previous calculations that a single stage rocket is limited by the payload and structural ratio as to its maximum speed capability. The structural ratio is one of the main contributors to this limit. However we can improve rocket performance by discarding the structural mass that is no longer required. This procedure is called staging. We an think of each stage as a single rocket. Then we can have a stage payload ratio, a stage structural ratio, and a structural fuel ratio. Furthermore, these ratios can be different for each stage. However, the key that links the stages together is that the payload of the n^{th} stage is the $n+1^{th}$ stage. If the vehicle has N stages, then the $N+1^{th}$ stage is the final payload. If we assume that all the stages have the same exhaust velocities, then the final velocity, or burnout velocity is equal to the sum of the stage burnout velocities:

$$V_{final} = \sum_{k=1}^{N} V_{e_k} \ln \frac{m_{1_k}}{m_{2_k}} \tag{29}$$

The overall payload ratio is the product of each stage payload ratios:

$$\lambda_{p_{overall}} = \prod_{k=1}^{n} \lambda_{p_k} \tag{30}$$