2. Preliminaries

In order to evaluate the terms in the differential equations of motion, it is necessary that discuss the issue of units and dimensions. These ideas are fundamental for all analysis and must be understood in order to obtain qualitative results. In all problems, units and dimensions must be consistent.

Dimensions: Dimensions are qualitative in nature and designate a type of unit, not related in any way to a value. In this course we will use four fundamental dimensions:

[M] mass
[L] length
[T] time
[θ] temperature

such a system is usually called MLT system of dimensions. Note that the dimension [F] force ***is not included***. The force dimension can be derived from the three MLT dimensions using Newton's second law:

$$F = m\,a$$
$$[F] = [M][L\,T^{-2}] = [MLT^{-2}] \tag{1}$$

For example, we can express pressure, Force/unit area with the following dimensions:

$$[P] = [FL^{-2}] = [MLT^{-2}L^{-2}] = [ML^{-1}T^{-2}]$$

(Note; It is possible to build a system on the dimensions FLT with M being the derived one. However that system will not be use here. The one use here is considered the more fundamental system).

Units: Units are qualitative in nature and will reflect a specific amount. There are several systems of units, but we will be concerned with only two: The US Customary and The Systeme International (SI). Unfortunately both are frequently used, with the US system primarily used in the US, and the SI system used throughout the world and in the US. The units and dimensions associated with these systems is given below:

Dimension	US	SI
[M]	slug (slug)	kilogram (kg)
[L]	foot (ft)	meter (m)
[T]	second (sec)	second (s)
[θ]	deg Rankine (deg Fahrenheit)+459.688)	deg Kelvin (deg Centigrade+273.16)
[F] (derivable)	pound (lb)	Newton (N)

Clearly, from Newton's second law:

$$1\ \text{lb} = 1\ \text{slug} \cdot 1\ \text{ft/sec}^2 \qquad\qquad 1\ \text{N} = 1\ \text{kg} \cdot 1\ \text{m/s}^2$$

These are called ***proper units*** since the force is directly related to the mass and acceleration, without any proportionality factors. Note that under no circumstances does the mass unit lb_m appear anywhere. It is not a proper unit and will not appear in the US or SI system of units!

Since we will deal with both sets of units, we need conversion factors and constants.

Constants:

Constant	US	SI
Perfect Gas Constant R	$1716.488 \dfrac{\text{ft lb}}{\text{slug °R}}$	$287.0368 \dfrac{\text{Joules}}{\text{kg °K}}$
Gravitational Constant g_0	$32.174 \dfrac{\text{ft}}{\text{sec}^2}$	$9.807 \dfrac{m}{s^2}$

Conversions:

1 ft	=	0.3048 m	Units of length
1 ft^2	=	0.0929 m^2	Units of area
1 slug/ft^3	=	515.379 kg/m^3	Units of density
1 ft-lb	=	1.3558 Nm (Joules)	Units of energy
1 lb	=	4.4482 N	Units of force
1 slug	=	14.5939 kg	Units of density
1 HP	=	745.6999 Watts	Units of power
1 lb/ft^2	=	47.88026 N/m^2 (Pascal)	Units of Pressure
1 ft^3	=	0.02832 m^3	Units of Volume

In addition to the fundamental units, we often get information in terms of nautical miles and knots. The conversions for these (non-fundamental units) are:

		US		SI		
1 nautical mile	=	6076.1 ft	=	1852.0 m		
1 nautical mile/hour	=	1 knot				
1 knot	=	1.6878 ft/sec	=	0.5144 m/s	=	1.1508 miles/hour

These last values can be calculated from the basic unit relations given previously:

$$\frac{1 \text{ nautical mile}}{\text{hr}} \cdot \frac{1 \text{ hr}}{3600 \text{ sec}} \cdot \frac{6076.1 \text{ ft}}{1 \text{ nautical mile}} = 1.6878 \frac{\text{ft}}{\text{sec}}$$

and

$$1.6878 \frac{\text{ft}}{\text{sec}} \cdot \frac{1852.0 \text{ m}}{6076.1 \text{ ft}} = 0.5144 \text{ m/s}$$

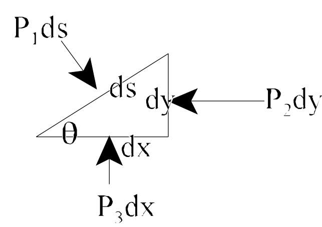
Fluid Mechanics

We need to define some properties of fluids (air), that will be important to us at a later time. These include pressure, density, temperature, viscosity, and the speed of sound.

Pressure: Pressure in general is defined as a ***force per unit area***. Hence the force on any body is the pressure times the area. If we consider a triangular shaped chunk of fluid in static equilibrium, we can sum the forces on it:

$$\Sigma F_x = P_1 ds \sin\theta - P_2 dy = 0$$

$$\Sigma F_z = P_3 dx - P_1 ds \cos\theta = 0$$



But:

$$dx = ds \sin\theta \text{ and } dy = ds \cos\theta$$

Therefor, we see that $P_1 = P_2 = P_3$. That is that the pressure is the same at a given point in a fluid, and that the pressure is a ***property of the fluid*** at that point. Pressure is a scalar, and has the dimensions: $[P] = [FL^{-2}]$ with the associated units: lb/ft^2 or N/m^2.

We can also describe the (scalar) pressure as the following limit:

$$P = \lim_{dA \to 0} \frac{dF}{dA} \tag{2}$$

Density: Density is defined as the mass per unit volume. We can define it as we take a particular volume and keep shrinking it until we get a point of fluid. The density then is

$$\rho = \lim_{d\forall \to 0} \left(\frac{dm}{d\forall} \right) \tag{3}$$

Density is a scalar, and has the dimensions: $\left[ML^{-3}\right]$ with the corresponding units: slug/ft^3, kg/m^3.

Incompressible flow: Density is a constant

Compressible flow: Density depends on pressure and temperature through the **perfect gas law** (see below)**.**

Temperature: The temperature is a point property that measures the mean molecular kinetic energy. Temperature is a scalar with dimension; $[\,\theta\,]$ and has the units deg R, deg K. Sometimes we use the Fahrenheit or the centigrade scale, but not when dealing with fluid (air) calculations.

Perfect Gas Law: The perfect gas law is a rule that tells us how the pressure and temperature of a perfect gas are related. This law is given by the following relation:

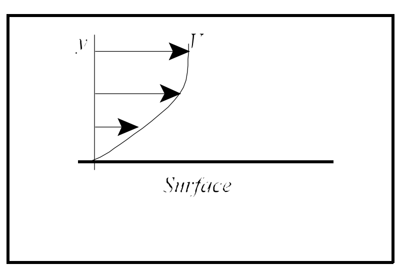
$$P = \rho R T \tag{4}$$

Here, P = pressure
ρ = density
R = Perfect Gas Constant

The definition of a perfect gas is beyond the level of this discussion. Suffice to say that air behaves like a perfect gas.

Viscosity: Viscosity of a fluid is the property of the fluid that indicates how the fluid "drags" along a surface that is parallel to the flow. The force per unit area caused by this fluid motion (not pressure) is called the ***shear stress***. This shear stress is proportional to how rapidly the velocity changes at the surface with displacement away from the surface, the so-called velocity gradient. The constant of proportionality is called the viscosity.
We have:

$$\tau = \mu \frac{dV}{dy} \tag{5}$$



where τ = shear stress
μ = viscosity

The dimension of the viscosity can be determined from its definition:

$$\left[FL^{-2}\right] = \left[\mu\right]\left[\frac{LT^{-1}}{L}\right] = \left[\mu\right]\left[T^{-1}\right]$$

Then solving for $\left[\mu\right]$(recall that [F] = [MLT^{-2}]):

$$\left[\mu\right] = \left[ML^{-1}T^{-1}\right] \text{with the corresponding units: } \frac{\text{slug}}{\text{ft sec}}, \quad \text{or} \quad \frac{\text{kg}}{\text{m s}}$$

Speed of Sound: The speed of sound is just that, the speed at which sound (or pressure waves) would travel through a fluid. It is related to the compressibility properties of the fluid. For example water is barely compressible, and the speed of sound is very fast. Air is more compressible and the speed of sound is slower than water. A measure of how the compressibility of the fluid affects an analysis is by the value of the ***Mach number***, the ratio of the velocity to the speed of sound. Flows at low Mach numbers can be treated as incompressible, while those at high Mach numbers must be treated as compressible. What we mean by high or low is yet to be determined.

Steady Flow

Flow fields can be considered to be either steady, or unsteady. Most of our considerations will be applied to a steady flow situation.

Steady flow: Steady flow is defined as flow where the properties of the fluid at a given point ***do not change in time.***

Unsteady flow: Unsteady flow is flow that is not steady. That is the properties at a given point in the fluid change in time. (Such as density, temperature, pressure, etc)

Note: The concept of steady or unsteady flow can depend on from where the flow is being observed. For our purposes we are interested in observing the flow relative to an aircraft. If we observe from the ground, as the aircraft goes by, the properties of the air change with the passing of the aircraft, hence the flow appears to us as being unsteady. However if we sit on the wing of the aircraft, (and the aircraft is flying at constant speed and altitude) then the flow as we see it is steady. This is the point of view we must take. ***So rather than considering the air to be still and the aircraft flying through it, we will consider the aircraft to be still, and the air flowing over***

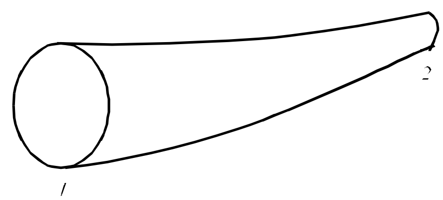
it, just like in a wind tunnel. This idea must be fully understood or you can get strange results when using the equations to be developed later!

Consequences of Steady Flow

By definition, in steady flow, the properties of the fluid at a point do not change in time. Hence the velocity at a particular point in steady flow does not change in time. Therefore, if we track a particle of fluid as it moves through a fluid (steady flow), it will trace out a path that will be the same for any other particle of fluid that is on that path. This path is called a ***streamline*** that will always be fixed in space. The velocity is always tangent to the streamline and hence ***no fluid can cross a streamline***. Although true for unsteady flows, this fact is the most useful when applied to steady flows since in unsteady flows the streamline could be changing, while for steady flows it is fixed in space.

If we take a bunch of streamlines ***in steady flow***, we can form a stream tube, a tube whose sides are made up of streamlines. Since no fluid can cross a streamline, what flows in one end of the tube must flow out the other end! We have the following figure:

Here we see "stuff" flows in through one end, and then out the other end, and nothing can "leak" out the side. Therefore we can introduce the idea of continuity in the form of the steady continuity equation.



Stream tube

Continuity (1-D):

We can develop the ***one-dimensional*** continuity equation that tells us the mass that flows in one end, must flow out the other end. If we designate the mass flow rate as $\dot{m}$, then the ***general 1-D continuity equation*** for ***steady flow*** becomes:

$$\boxed{\dot{m} = \rho A V = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad (6)}$$

where $\dot{m}$ = mass flow rate with units (slugs/sec) or (kg/s)]
ρ = density of the fluid
A = the cross sectional area (measured perpendicular to $\vec{V}$
V = the velocity (speed) of the fluid

The subscripts 1 and 2 designate any two different stations at which measurements are taken.

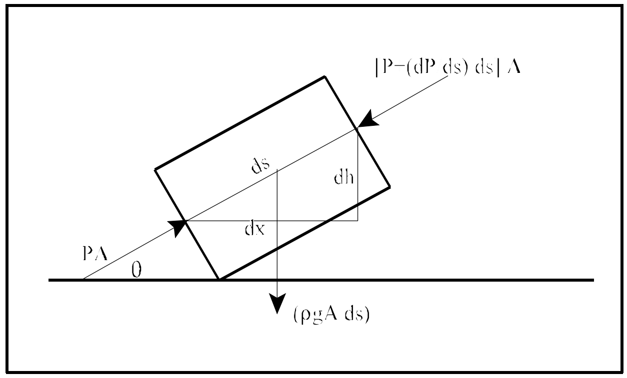
If the flow is incompressible, the density does not change and hence the ***general***

incompressible 1-D continuity equation for ***steady flow*** becomes:

$$A\,V = A_1 V_1 = A_2 V_2 \tag{7}$$

Where ***A V*** is the volume flow rate and has units (ft^3/sec) or (m^3/s).

We would now like to write an equation of motion for a chunk of fluid moving along a streamline. To do this we will sum the forces acting on the chunk of fluid and set it equal to the mass times the acceleration. First lets draw the free-body diagram!



The forces perpendicular to the straight streamline must sum to zero, and are not shown. If the sum were not zero, then there would be an acceleration normal to the streamline and the streamline would be curved. We will assume that no such forces occur and that the streamline is straight.

We can sum the forces along the streamline and apply Newton's law in the streamline direction:

$$\Sigma F_s = m\,a_s :$$

$$PA - \left(P + \frac{dP}{ds}ds\right)A - \rho g A\,ds\sin\theta = \rho A\,ds\frac{dV}{dt}$$

$$PA - PA - \frac{dP}{ds}A\,ds - \rho\,g A\,dh = \rho\,A\,V dV$$

$$-dP - \rho\,g\,dh - \rho\,V dV = 0$$

Rearranging, we have the
<u>Differential Form of Bernoulli's Equation:</u>

$$dP + \rho\,V dV + \rho\,g\,dh = 0$$
$$\frac{dP}{\rho} + V dV + g\,dh = 0 \tag{8}$$

The relations developed in this section are the starting points for several developments to be given later. These include the standard atmosphere, manometers, and airspeed indicators. We will develop the integrated form of Bernoulli's equation for the both the case for incompressible flow (the usual development) and for compressible flow. The starting point for all these developments is Eq. (8).