Aircraft Dynamics

In order to discuss dynamic stability we essentially need to solve the differential equations of motion. However, before jumping into the full blown problem of aircraft motion, it is useful to look at some approximations first, starting with the simplest mathematical model and build up to the general case.

Roll motion approximation

I turns out that the rolling motion can be approximated by considering only the roll equation of motion. This equation assumes that the aircraft is pinned along its x or roll axis. It turns out that this approximation is fairly decent and gives a good approximation of the motion. We have already used it when we discussed the steady state roll rate earlier. The roll equation of motion and the associated kinematic equation relating roll rate with the roll angle take the form:

$$\begin{aligned} L &= I_x \dot{p} \\ \dot{\phi} &= p \end{aligned} \tag{1}$$

We now would like to examine the motion in the neighborhood of some steady state reference motion. Just like our discussion of static stability, the dynamic motion must be examined with respect to a reference motion, usually steady state. If we consider all variables whose derivatives appear as "state" variables, then steady state motion is that which can occur when these derivatives are zero. Here, looking at Eq. (1), we can see that our steady state reference flight condition is given by:

$$\begin{aligned} L_{ref} &= 0 \\ p_{ref} &= 0 \end{aligned} \tag{2}$$

We can also note that the aerodynamic rolling moment is not a function of the roll angle, ϕ, and therefore ϕ does not appear in the roll equation of motion. Consequently if we are only interested in the roll rate, we can drop the ϕ equation and consider it "ignorable". Under these circumstances we could have a reference flight condition such as $p = p_{ref} \neq 0$. So all the results that we develop can be considering the change in roll rate from rest, or from a reference roll rate. For the assumptions made here, we can consider the roll moment to have the following functional dependence:

$$L = L(p, \delta_a) \tag{3}$$

We now seek to linearize the roll equation of motion. A convenient way to carry this out is to put all the derivative terms on the left, and the remaining terms on the right. For our simple problem this activity leads to (we will drop the ϕ equation for reasons stated earlier),

$$\dot{p} = \frac{1}{I_x} L(p, \delta_a) \tag{4}$$

To examine the properties in the neighborhood of our reference flight condition, we will let the variables be equal to their values in the reference flight condition, plus an additional small change or perturbation. Hence we have:

$$p = p_{ref} + \Delta p \qquad \delta_a = \delta_{a_{ref}} + \Delta\delta_a \tag{5}$$

If we substitute these quantities into Eq. (4) we get the following:

$$\begin{aligned} \dot{p}_{ref} + \Delta\dot{p} &= \frac{1}{I_x} L(p_{ref} + \Delta p, \delta_{a_{ref}} + \Delta\delta_a) \\ &= \frac{1}{I_x}\left[L(p_{ref}, \delta_{a_{ref}}) + \left.\frac{\partial L}{\partial p}\right|_{ref} \Delta p + \left.\frac{\partial L}{\partial \delta_a}\right|_{ref} \Delta\delta_a + \cdots \right] \end{aligned} \tag{6}$$

Since $\dot{p}_{ref}$ and $L(p_{ref}, \delta_{a_{ref}}) = 0$, the first terms on each side of the equations are zero and are dropped. What remains is a first order ordinary differential equation in the change in the roll rate:

$$\begin{aligned} \Delta\dot{p} &= \frac{1}{I_x}\left[L_p \Delta p + L_{\delta_a} \Delta\delta_a \right] \\ &= \frac{L_p}{I_x} \Delta p + \frac{L_{\delta_a}}{I_x} \Delta\delta_a \end{aligned} \tag{7}$$

where $L_p = \frac{\partial L}{\partial p}$, and $L_{\delta_a} = \frac{\partial L}{\partial \delta_a}$ and are called dimensional stability derivatives. (Note that there are other definitions, for example some folks define $L_p = \frac{1}{I_x}\frac{\partial L}{\partial p}$, etc., that is they include the moment of inertia in the term. It makes the equations look simpler, but can lead to confusion so it won't be used here. It is, however, used in Etkin and Reid).

Computing the dimensional stability derivative

Since $L = C_l \bar{q} S b$, we can simply take the derivative with respect to the variable of interest:

$$\frac{\partial L}{\partial p} = L_p = \frac{\partial C_l}{\partial p} \bar{q} S b \cdot \frac{b/2V}{b/2V} = \frac{\partial C_l}{\partial \frac{pb}{2V}} \frac{\bar{q} S b^2}{2V} = C_{l_p} \frac{\bar{q} S b^2}{2V} \tag{8}$$

and

$$\frac{\partial L}{\partial \delta_a} = L_{\delta_a} = C_{L_{\delta_a}} \bar{q} S b \tag{9}$$

Uncontrolled Response to Initial Disturbance

If we hold the control fixed at its reference value, the $\Delta\delta_a = 0$, and the equation of motion takes the form:

$$\Delta \dot{p} = \frac{L_p}{I_x} \Delta p \tag{10}$$

One method to solve this equation is to assume a solution of the form $\Delta p = Ce^{\lambda t}$ with the derivative, $\Delta \dot{p} = C\lambda e^{\lambda t}$. If we substitute this "solution" back into the differential equation, Eq. (10) , divide through by $Ce^{\lambda t}$ (which can not be zero if we want a non-trivial solution) we are left with the result that in order for our assumed solution to be correct solution to Eq. (10), then it is necessary that λ take on a particular value determined from the ***characteristic equation***,

$$\lambda - \frac{L_p}{I_x} = 0$$

$$\lambda = \frac{L_p}{I_x} \tag{11}$$

The solution to the original equation is then given by:

$$\Delta p(t) = C e^{\frac{L_p}{I_x} t} \tag{12}$$

We can evaluate the constant C by specifying initial conditions, at $t = t_0$, $\Delta p = \Delta p(0) = \Delta p_0$. Substituting into the solution, Eq. (12) and doing some algebra, gives the result,

$$\Delta p(t) = \Delta p_0 e^{\frac{L_p}{I_x}(t - t_0)} \tag{13}$$

For most problems, the initial time is zero, $t_0 = 0$, and Eq. (13) simplifies accordingly.

Equation (13) describes the motion after an initial roll rate disturbance is introduced into the system. An ideal behavior would be one that goes to zero as time goes to infinity. That would mean that the disturbance would die out and the system would return to the reference flight condition, in this case to the reference roll rate (typically = 0). It is clear by looking at the solution, and noting that $I_x > 0$, that the disturbance will die out only if $L_p < 0$. Hence in general we can say that a first order system is dynamically stable if its characteristic value, λ, is less than zero. The motion is a damped roll rate and this motion is called ***rolling convergence***.

As an example, consider an aircraft that has S = 230 ft^2, b = 34 ft, $\bar{q} = 134.6$ lbs/ft^2 (V = 677 ft/sec) and an x moment of inertia, $I_x = 28000$ slug ft^2. In addition, the non-dimensional stability derivative is $C_{l_p} = -0.45$. Then

$$\frac{L_p}{I_x} = C_{l_p}\frac{\bar{q}Sb^2}{2V} = (-0.45)\frac{134.6(230)34.0^2}{2(677.0)} = -0.424 \text{ /sec}$$

and

$$\Delta p(t) = \Delta p_0 e^{-0.424t}$$

Properties of First Order Solution

All the information regarding the properties of the first order response is carried in the characteristic value, λ. For the sake of discussion, we can assume the value of $\lambda = a$. Then the solution is given by

$$\Delta p = \Delta p_0 e^{at} \tag{14}$$

Then the following properties can be determined:

1) The system is dynamically stable (the disturbance goes to zero in time) if a < 0.

2) The time to half amplitude of the initial disturbance is

$$\frac{1}{2} = e^{at_{1/2}} \qquad \boxed{t_{1/2} = \frac{\ln 2}{|a|}}$$

3) The "time constant" for this system is

$$\boxed{\tau = \frac{1}{|a|}}$$

If we apply these measures to our example problem we obtain the following values:

$$t_{1/2} = \frac{\ln 2}{0.424} = 1.63 \text{ sec} \qquad \tau = \frac{1}{0.424} = 2.358 \text{ sec}$$

The significance of the time constant can be determined by re-examining the solution. If we write it in terms of the time constant τ, we have:

$$\frac{\Delta p(t)}{\Delta p_0} = e^{-\frac{t}{\tau}} \tag{15}$$

We can then make a plot (or table) of the ratio of the response to initial displacement vs t/τ (or equivalently measure time in time constants, i.e. 1 time constant, 2 time constants, etc.

Here we see that if we displace the vehicle in roll, that it will return to within 5% of the original displacement in 3 time constants and to within 2% in 4 time constants. So we might say that the vehicle returns to its reference state in 4 time constants. Hence the time constant is a measure of performance, a short time constant it behave rapidly, and a long time constant, it behave sluggishly.

t/τ	$\Delta p/\Delta p_0$
0	1
1	0.368
2	0.135
3	0.0497
4	0.0183

Response to Control Input

If we now displace the control surface, we can solve the differential equation with the specified control input. Under these circumstances, the equation of motion becomes

$$\Delta \dot{p}(t) = \frac{L_p}{I_x} \Delta p(t) + \frac{L_{\delta_a}}{I_x} \Delta \delta_a(t) \tag{16}$$

In order to solve this equation we need to first specify the control input. For our purposes here, we will specify a step in put to the control. That is it is zero until time t_0 (=0), and is a constant value for all $t > t_0$, $\Delta \delta_a = A$ = constant. Therefor for times greater than zero the equation of motion looks like:

$$\begin{aligned} \Delta \dot{p}(t) &= \frac{L_p}{I_x} \Delta p(t) + \frac{L_{\delta_a}}{I_x} A \\ &= a \Delta p(t) + bA \end{aligned} \tag{17}$$

where a and b are defined appropriately.
The solution to this equation consists of a homogeneous solution plus a particular solution. The homogenous solution is in the same form as the uncontrolled solution presented earlier. The particular solution can be obtained by assuming a solution that has the same form as the input. In this case the particular solution assumed is a constant, $\Delta p = D = const$. Substituting this value into Eq. (17) we get

$$0 = aD + bA \qquad \Rightarrow \qquad D = -\frac{b}{a}A \tag{18}$$

so that the total solution is given by

$$\Delta p(t) = C e^{at} - \frac{b}{a}A \tag{19}$$

From initial conditions we can find the value of the constant C. At $t = 0$, $\Delta p(0) = \Delta p_0$ and C turns out to be $C = \frac{b}{a}A + \Delta p_0$. Combining these results gives the final solution for the response to a step of amplitude A:

$$\begin{aligned} \Delta p(t) &= -A\frac{b}{a}\left[1 - e^{at}\right] + \Delta p_0 e^{at} \\ &= -A\frac{C_{l_{\delta_a}}}{C_{l_p}}\left[1 - e^{\frac{L_p}{I_x}t}\right] + \Delta p_0\, e^{\frac{L_p}{I_x}t} \end{aligned} \tag{20}$$

If we assume that the initial roll rate is zero and note that as $t \to \infty$, $\Delta p \to \Delta p_{ss}$, the steady state roll rate, we can write the solution to the step input as

$$\Delta p(t) = \Delta p_{ss}\left[1 - e^{\frac{L_p}{I_x}t}\right] \tag{21}$$

If we compare Eq. (21) with Eq. (15) it is not difficult to come to the conclusion that if we put in a step aileron input, we will get to 98% of the steady state roll rate in 4 time constants! Hence the time constant is a measure of the time it takes to get to any specified steady state roll rate. Note that it takes the same time to get to 98% of 360 degrees/sec as to get to 98% of 90 degrees/sec (note that the aileron deflection would be less to maintain a 90 deg/sec steady state roll rate).

The result we obtained for roll motion are approximate. However this approximation is generally a good one.

Pure Yaw Motion

The next level of difficulty in describing vehicle dynamics is to consider an aircraft in a wind tunnel pinned along the z axis so that it is free to yaw. The equation of motion is given by

$$N = I_z \ddot{\psi} \tag{22}$$

We can consider the reference flight condition as the steady state condition with $N_{ref} = 0$. Further we can assume that the yaw moment is a function of sideslip angle, yaw rate, and rudder deflection, $N = N(\beta, \dot{\beta}, r, \delta_r)$. In addition, we can note that in the wind tunnel a unique relation exists between the heading angle, ψ, and the sideslip angle, β. Namely, $\psi = -\beta$ and $\dot{\psi} = -\dot{\beta}$. We can now make the same substitutions that we did in the roll equation. Let the variables take on the values of the reference conditions plus a small disturbance. We have: $\ddot{\psi}_{ref} + \Delta\ddot{\psi} = -(\ddot{\beta}_{ref} + \Delta\ddot{\beta})$, $\dot{\psi} + \Delta\dot{\psi} = -(\dot{\beta}_{ref} + \Delta\dot{\beta}) = r_{ref} + \Delta r$. We can write the yaw moment equation as:

$$\ddot{\psi}_{ref} + \Delta\ddot{\psi} = \frac{1}{I_z}\left[N_{ref} + \frac{\partial N}{\partial \beta}\delta\beta + \frac{\partial N}{\partial \dot{\beta}}\Delta\dot{\beta} + \frac{\partial N}{\partial r}\Delta r + \frac{\partial N}{\partial \delta_r}\Delta\delta_r\right] \tag{23}$$

Noting that all the reference conditions are zero, and making some of the substitutions indicated previously, we can rewrite the Eq. (23) in terms of β in the following manner:

$$\Delta\ddot{\beta} - \left(\frac{N_r}{I_z} - \frac{N_{\dot{\beta}}}{I_z}\right)\Delta\dot{\beta} + \frac{N_\beta}{I_z}\Delta\beta = \frac{N_{\delta_r}}{I_z}\Delta\delta_r \tag{24}$$

For the uncontrolled case we have:

$$\Delta\ddot{\beta} - \left(\frac{N_r}{I_z} - \frac{N_{\dot{\beta}}}{I_z}\right)\Delta\dot{\beta} + \frac{N_\beta}{I_z}\Delta\beta = 0 \tag{25}$$

We can compare this with the standard form for a second order ordinary differential equation:

$$\Delta\ddot{x} + 2\zeta\omega_n\Delta\dot{x} + \omega_n^2\Delta x = 0 \tag{26}$$

We can write the original equation in the form $\Delta\ddot{\beta} + b\Delta\dot{\beta} + c\Delta\beta = 0$. It is clear that the equation is completely described by the coefficients b and c. If we look at the “standard” form, we that it is also completely described by the two parameters ζ, and ω_n. It turns out that these two parameters are more useful than b and c in describing the characteristics of the solution to this equation and hence are used more often. The relations are clear:

$$\zeta = \frac{b}{2\sqrt{c}} = -\frac{N_r - N_{\dot{\beta}}}{2\sqrt{I_x N_\beta}} \qquad \omega_n = \sqrt{c} = \sqrt{\frac{N_\beta}{I_x}} \tag{27}$$

The Characteristic Equation and Characteristic values (eigenvalues)

The utility of these parameters becomes more clear when we seek a solution to the second order ordinary differential equation, Eq. (25) or Eq. (26). Again, we assume a solution of the form $x = Ce^{\lambda t}$. Then $\dot{x} = C\lambda e^{\lambda t}$, and $\ddot{x} = C\lambda^2 e^{\lambda t}$. If we substitute these into the differential equation and divide through by $Ce^{\lambda t}$ (which is non zero) we end up with:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{28}$$

Eq. (28) is the characteristic equation and gives us the values of λ for which are assumed solution is the solution. That is, $x = Ce^{\lambda t}$ is a solution to Eq. (26) only if λ satisfies Eq. (28). The solution to Eq. (28) is given by:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{29}$$

There are ***three possible cases*** that must be treated:

$\zeta > 1$ (Over damped) For this case there are two real roots given by Eq. (29). Hence the solution is given by:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{30}$$

where C_1 and C_2 are determined from initial conditions at $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, to give:

$$x(t) = \left[\frac{\lambda_2 x_0 - \dot{x}_0}{\lambda_2 - \lambda_1}\right] e^{\lambda_1 t} + \left[\frac{-\lambda_1 x_0 + \dot{x}_0}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \tag{31}$$

The important feature to note here is that x(t) behaves like two first order systems, each with its own time constant, and each with its own time to half amplitude. If we want to consider the system properties, then we take the properties associated with the **longest** time constant and time to half amplitude as the system properties.

$$\tau_{system} = \text{greater of} \begin{cases} \tau_1 = \dfrac{1}{|\lambda_1|} \\ \tau_2 = \dfrac{1}{|\lambda_2|} \end{cases} \tag{32}$$

$\zeta = 1$ (Critically damped) For this case there are two real roots that are equal:

$$\lambda_{1,2} = -\zeta\omega_n = -\omega_n \tag{33}$$

The corresponding solution is

$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} = x_0 e^{\lambda t} + (\dot{x}_0 - \lambda x_0) e^{\lambda t} \tag{34}$$

The important thing to note here is that if the solution is negative, $x(t) \to 0$ as $t \to \infty$

$\zeta < 1$ (Under-damped case) For this case the solution is a complex conjugate pair:

$$\begin{aligned} \lambda_{1,2} &= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \\ &= n \pm i\omega \end{aligned} \tag{35}$$

where $n = -\zeta\omega_n$ and is the real part, and $\omega = \omega_n\sqrt{1-\zeta^2}$, is the imaginary part.

The solution to the differential equation is given by:

$$\begin{aligned} x(t) &= e^{nt}\left(C_1\cos\omega t + C_2\sin\omega t\right) \\ &= e^{nt}\left(x_0\cos\omega t + \frac{\dot{x}_0 - nx_0}{\omega}\sin\omega t\right) \end{aligned} \tag{36}$$

In all cases, it is clear that if the ***real part of the characteristic value is negative***, ***the system is dynamically stable*** and $x(t) \to 0$ as $t \to \infty$!

For the last case, the motion is oscillatory with decreasing amplitude in time if it is stable, and increasing amplitude in time if it is not stable. We can characterize these motions by the time to double or half amplitude, the period, and a time constant in the following manner:

$$t_{1/2} = \frac{\ln 2}{|n|} \qquad T_p = \frac{2\pi}{\omega} \qquad \tau = \frac{1}{|n|} \tag{37}$$

We can also define a cycles to half amplitude:

$$N_{1/2} = \frac{t_{1/2}}{T_p} = \frac{\omega \ln 2}{2\pi n} \tag{38}$$

Longitudinal Pinned Aircraft

We can repeat the above exercise for a vehicle pinned in the wind tunnel along the y axis so that it is free to pitch. If we do that we note the following relations: $\theta = \alpha$, $\dot{\theta} = \dot{\alpha} = q$. We can write the pitch equation of motion and assume a functional form for the pitch moment:

$$I_y \ddot{\theta} = M(\alpha, \dot{\alpha}, q, \delta_e) \tag{39}$$

Proceeding as we did for the yaw moment we can arrive at the following second order ordinary differential equation:

$$\Delta\ddot{\alpha} - \frac{1}{I_y}\left(M_q + M_{\dot{\alpha}}\right)\Delta\dot{\alpha} - \frac{M_\alpha}{I_y}\Delta\alpha = \frac{M_{\delta_e}}{I_y}\Delta\delta_e \tag{40}$$

Again we can set the control input to zero and just consider the response to initial conditions.

$$\Delta\ddot{\alpha} - \frac{1}{I_y}\left(M_q + M_{\dot{\alpha}}\right)\Delta\dot{\alpha} - \frac{M_\alpha}{I_y}\Delta\alpha = 0$$
$$\Delta\ddot{\alpha} + 2\zeta\omega_n\Delta\dot{\alpha} + \omega_n^2\Delta\alpha = 0 \tag{41}$$

By comparing coefficients we can see that:

$$\zeta = -\frac{M_q + M_{\dot{\alpha}}}{2\sqrt{-I_y M_\alpha}} \qquad \omega_n = \sqrt{-\frac{M_\alpha}{I_y}} \tag{42}$$

An Example: Consider a vehicle that has the characteristics: W = 636600 lbs, I_y = 33.1 x10^6 slug ft^2, S = 5500 ft^2, b = 195.68 ft, $\bar{c}$ = 27.31 ft, $\bar{q}$ = 92.697 lbs/ft^2, $\bar{q}S$ = 509834 lbs, and V = 279.1 ft/sec. We can calculate the appropriate coefficients from the non-dimensional coefficients given: $C_{m_\alpha} = -1.26$, $C_{m_q} = -20.8$, and $C_{m_{\dot{\alpha}}} = -3.2$.

$$M_\alpha = C_{m_\alpha} \bar{q} S b = -1.26\,(509834)27.31 = -17.5437x10^6 \text{ ft lbs/rad} = -3.0619x10^5 \text{ ft lbs/deg}$$

$$\frac{M_\alpha}{I_y} = \frac{-17.5437x10^6}{33.1x10^6} = 0.530 \text{ /sec}^2$$

$$M_q = C_{m_q} \frac{\bar{q} S \bar{c}^2}{2V} = -20.8 \frac{509834\,(27.31)^2}{2(279.1)} = -14169212 \text{ ft lbs/rad}$$

$$\frac{M_q}{I_y} = \frac{-14169212}{33.1x10^6} = -0.4281 \text{ /sec}$$

$$\frac{M_{\dot\alpha}}{I_y} = C_{m_{\dot\alpha}} \frac{\bar{q} S \bar{c}^2}{I_y 2V} = -3.2 \frac{509835\,(27.31^2)}{33.1x10^6\,(2)\,279.1} = -0.0659 \text{ /sec}$$

The final equation becomes:

$$\Delta\ddot\alpha - (-0.4281 - 0.0659)\Delta\dot\alpha - (-0.530)\Delta\alpha = 0$$
$$\Delta\ddot\alpha + 0.494\Delta\dot\alpha + 0.530\Delta\alpha = 0$$

The characteristic equation is given by:

$$\lambda^2 + 0.494\lambda + 0.530 = 0$$

The characteristic values (eigenvalues) are given by:

$$\lambda = -0.247 \pm i\,0.685 \text{ /sec}$$
$$= n \pm i\omega$$

Hence $n = -0.247$ /sec, and $\omega = 0.685$ /sec. We can compute the properties of this oscillatory motion:

$$t_{1/2} = \frac{\ln 2}{|n|} = \frac{\ln 2}{0.247} = 2.800 \text{ sec} \qquad T_p = \frac{2\pi}{\omega} = \frac{2\pi}{0.685} = 9.175 \text{ sec} \qquad \tau = \frac{1}{0.247} = 4.048 \text{ sec}$$

We can also calculate other parameters from the coefficients directly:

$$\omega_n = \sqrt{c} = \sqrt{0.530} = 0.723 \text{ /sec} \qquad \zeta = \frac{b}{2\omega_n} = \frac{0.494}{2\,(0.723} = 0.339$$

Note that $\omega \neq \omega_n$, and that the period is based on ω and ***not*** ω_n! From the geometry of the complex number, and the parameters in the equations, we can arrive at some other relationships that occasionally are useful:

$$\omega_n = \sqrt{n^2 + \omega^2} \qquad \zeta = \frac{|n|}{\omega_n} = \frac{|n|}{\sqrt{n^2 + \omega^2}} \tag{43}$$

Finally, if we wanted to we could write the solution for the response to initial conditions $\Delta\alpha_0$, and $\Delta\dot{\alpha}_0$:

$$\Delta\alpha(t) = e^{-0.247t}\left(\Delta\alpha_0 \cos 0.685t + \frac{\Delta\dot{\alpha}_0 + 0.247\,\Delta\alpha_0}{0.685} \sin 0.685t \right)$$

Generally we are not interested in the actual motion, just in the characteristic of the motion, time to double or half amplitude, period, frequency, and time constants.