Stability and Control
Complete Vehicle Pitch Stability and Control

The equations for the complete vehicle lift and pitch-moment were developed previously and are repeated here:

$$\boxed{\begin{aligned} C_L &= C_{L_\alpha}\bar{\alpha} + C_{L_{\delta_e}}\delta_e \\ C_m &= C_{m_{0L}} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\delta_e \end{aligned}} \qquad (1)$$

where $\bar{\alpha} = \alpha - \alpha_{0L}$.

Longitudinal Static Stability Parameter

The indicator of longitudinal static stability (or static stability in pitch) is the pitch-moment curve slope:

$$\boxed{C_{m_\alpha} = a_{wb}\left[(h - h_{n_{wb}}) - \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}(h_{ht} - h)\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right]} \qquad (2)$$

where:

h is the non-dimensional distance (in mean aerodynamic chord lengths) of the cg behind the leading edge of the wing mean aerodynamic chord.

$h_{n_{wb}}$ is the non dimensional distance (in mean aerodynamic chord lengths) of the aerodynamic center of the wing-body behind the leading edge of the mean aerodynamic chord.

h_{ht} is the non-dimensional distance (in mean aerodynamic chord lengths) of the aerodynamic center of the horizontal tail behind the leading edge of the wing mean aerodynamic chord.

$\frac{\partial\epsilon}{\partial\alpha}$ Is the (average) change in downwash at the horizontal tail with angle-of-attack

We can consider the effect of the cg position (h) on the sign of C_{m_α}. If the cg is well forward, the value of h is small and could even be negative (in front of the wing mean aerodynamic chord). In this case the first term on the right-hand side of the equation, $(h - h_{n_{wb}})$, is negative, and second term is also negative. Hence the vehicle is statically stable. As we move the center-of-gravity aft, the value of h gets larger, and eventually the first term goes to zero, and the magnitude of the second term becomes smaller. However the value of C_{m_α} still remains negative, and the vehicle stable. If we move the cg further aft, the first term becomes positive and increases in magnitude, while the second term remains negative, but decreases in magnitude, $(h_{ht} - h)$, gets smaller. Eventually there is some position where the value of C_{m_α} becomes zero. If the cg were located at that position, the vehicle would be considered to be neutrally stable. That location is called the ***neutral point***. Be definition, it is also the ***aerodynamic center of the aircraft***. Hence if the cg is located in front of the neutral point, the vehicle is statically stable, and if the cg is located aft of the neutral point, the vehicle is statically unstable. Consequently locating the neutral point is of considerable interest.

Neutral Point Location

The neutral point location can be determined from Eq. (2) by setting it equal to zero and solving for $h \equiv h_n$, the neutral point. The result can be put in two forms:

$$h_n = \frac{h_{n_{wb}} + \dfrac{a_{ht}}{a_{wb}} \eta_{ht} \dfrac{S_{ht}}{S} h_{ht} \left(1 - \dfrac{\partial \epsilon}{\partial \alpha} \right)}{1 + \dfrac{a_{ht}}{a_{wb}} \eta_{ht} \dfrac{S_{ht}}{S} \left(1 - \dfrac{\partial \epsilon}{\partial \alpha} \right)} = h_{n_{wb}} + \frac{a_{ht}}{a} \eta_{ht} \frac{S_{ht}}{S} (h_{ht} - h_{n_{wb}}) \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \tag{3}$$

If we use Eq. (3) to eliminate $h_{n_{wb}}$ from Eq. (2), then we obtain the (obvious) result:

$$C_{m_\alpha} = a(h - h_n) \tag{4}$$

Equation (4) can be used to define the neutral point or the vehicle aerodynamic center. The distance between the center-of-gravity location and the neutral point is some measure of the level of stability that the vehicle has and is called the static margin. It is defined as:

Static Margin

$$K_n = h_n - h \tag{5}$$

and is positive if the vehicle is longitudinally aerodynamically statically stable.

Angle-of-Attack and Elevator Deflection Required For Equilibrium Flight

The requirements for aircraft control in the pitch plane can be determine from the two equations for the lift and pitch-moment.

$$\boxed{\begin{aligned} C_L &= C_{L_\alpha}\bar{\alpha} + C_{L_{\delta_e}}\delta_e \\ C_m &= C_{m_{0L}} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\delta_e \end{aligned}} \qquad (6)$$

For equilibrium (force and moment balance) we require that

$$\begin{aligned} \frac{W}{1/2\,\rho\, V^2 S} &= C_L = C_{L_\alpha}\bar{\alpha} + C_{L_{\delta_e}}\delta_e \\ 0 &= C_m = C_{m_{0L}} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\delta_e \end{aligned} \qquad (7)$$

Assuming we are given all the aerodynamic properties of the aircraft, all terms in Eq. (7) are known except for the angle-of-attack, $\bar{\alpha}$, and the elevator deflection, δ_e. Hence we have tow equations and two unknowns. Since the equations are linear in these variables, we can rearrange the equations and write them in a matrix form:

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta_e}} \\ C_{m_\alpha} & C_{m_{\delta_e}} \end{bmatrix} \begin{Bmatrix} \bar{\alpha} \\ \delta_e \end{Bmatrix} = \begin{Bmatrix} C_L \\ -C_{m_{0L}} \end{Bmatrix} \qquad (8)$$

We can invert the matrix and solve for the angle-of-attack and elevator to obtain:

$$\boxed{\bar{\alpha} = \frac{C_{L_{\delta_e}} C_{m_{0L}} + C_{m_{\delta_e}} C_L}{\Delta}} \qquad (9)$$

and

$$\boxed{\delta_e = -\frac{C_{L_\alpha} C_{m_{0L}} + C_{m_\alpha} C_L}{\Delta}} \qquad (10)$$

where Δ is the determinant of the coefficient matrix in Eq. (8) and is given by:

$$\boxed{\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{m_\alpha} C_{L_{\delta_e}}} \qquad (11)$$

Example 1: Given: $h = 0.3$ $h_{ht} = 2.5$ $\dfrac{\partial \epsilon}{\partial \alpha} = 0.4$ $S = 600\ \text{ft}^2$ $S_{ht} = 100\ \text{ft}^2$

$a_{wb} = 0.06/\text{deg}$ $a_{ht} = 0.04\ /\ \text{deg}$ $\eta_{ht} = 0.9$ $h_{n_{wb}} = 0.25$

Determine the stability of this aircraft and the location of the neutral point.

$$C_{m_\alpha} = a_{wb}\left[(h - h_{n_{wb}}) - \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$$

$$= 0.06\left[(0.3 - 0.25) - \frac{0.04}{0.06}(0.9)(2.5 - 0.3)\frac{100}{600}(1 - 0.4)\right]$$

$$= =0.0049\ /\text{deg} = -0.282\ /\text{rad}$$

Therefore the vehicle is stable!

$$a = a_{wb}\left[1 + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$$

$$= 0.06\left[1 + \frac{0.04}{0.06}(0.9)\frac{100}{600}(1 - 0.4)\right]$$

$$= 0.06\ [\ 1.06\ ]$$

$$= 0.0636\ \ /\text{deg}$$

The neutral is obtained from:

$$C_{m_\alpha} = a(h - h_n) = -0.0049 = 0.0636(0.3 - h_n) \Rightarrow h_n = 0.3770$$

and the static margin is $K_n = h_n - h = 0.3770 - 0.3 = 0.077$ $\Rightarrow$ **<u>Or 7.7% mac</u>**

Example 2: Given: $C_L = 0.419$ $C_{L_\alpha} = 2.2\ /\text{rad}$ $C_{m_\alpha} = -0.64\ /\text{rad}$ $C_{L_{\delta_e}} = 0.46\ /\text{rad}$

$C_{m_{\delta_e}}$ m= -1.24 /rad $C_{m_{0L}} = 0.05$

Define $\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{m_\alpha} C_{L_{\delta_e}} = 2.2\,(0.46) - (-0.64)\,(0.46) = -0.419\ \ /\text{deg}^2$

$$\bar{\alpha} = \frac{C_{m_{0L}} C_{L_{\delta_e}} + C_{m_{\delta_e}} C_L}{\Delta} = \frac{0.05\,(0.46) + (-1.24)(0.419)}{-2.433} = 0.204\ \text{rad} = 11.69\ /\text{deg}$$

$$\delta_e = -\frac{C_{L_\alpha} C_m 0L + C_{m_\alpha} C_L}{\Delta} = -\frac{2.2\,(0.05) + (-0.64)(0.419)}{-2.433} = -0.065\ \text{rad} = -3.724\ \text{deg}$$

Maximum Forward CG Position

As the cg position moves further forward, and the static margin increases, the range of elevator deflection required for equilibrium flight over a given range of speeds increases. Consequently as the cg moves forward, the elevator required for balancing the aircraft in slow flight becomes a large negative (up elevator) value. If the cg is far enough forward, the elevator will hit its limits at some relatively slow speed. We would like to be able to control the aircraft throughout its flight envelope and so it is desirable to be able to balance the vehicle in pitch at least through the stall speed. As a result, we would like to determine the cg position where the elevator hits the stop at the same time the aircraft stalls. This position would be the max forward limit of the cg. Anything further forward would not allow us to balance the aircraft at all speeds lower than stall. We can determine this location (approximately) from the following analysis.

The elevator required for balance is given by:

$$\delta_e = -\frac{C_{m_{0L}} C_{L_\alpha} + C_{m_\alpha} C_L}{C_{m_{\delta_e}} C_{L_\alpha} - C_{m_\alpha} C_{L_{\delta_e}}} \tag{12}$$

As we have done previously, we will designate the denominator term as,

$$\Delta = C_{m_{\delta_e}} C_{L_\alpha} - C_{m_\alpha} C_{L_{\delta_e}}$$

If we move the cg in the vehicle, the following terms change: C_{m_α}, and $C_{m_{\delta_e}}$. The remaining terms, $C_{m_{0L}}$, C_{L_α}, and $C_{L_{\delta_e}}$, do not change. However, even though terms in the denominator change, the complete denominator term, Δ, is a constant. Thus only C_{m_α} that appears in the denominator will change with movement of the cg. We can write the above equation in the form:

$$\delta_e = -\frac{C_{m_{0L}} C_{L_\alpha}}{\Delta} - \frac{C_{m_\alpha} C_L}{\Delta}$$

for an arbitrary cg position, and by definition of the max forward cg position we have:

$$\delta_{e_{\min}} = -\frac{C_{m_{0L}} C_{L_\alpha}}{\Delta} - \frac{C_{m_\alpha} C_{L_{\max}}}{\Delta}$$

where we are at max up elevator when the aircraft stalls at $C_{L_{\max}}$. We can solve for C_{m_α} to give us the following results:

$$C_{m_\alpha} = a(h_{\min} - h_n) = -\frac{\Delta}{C_{L_{\max}}}\left[\frac{C_{m_{0L}} C_{L_\alpha}}{\Delta} + \delta_{e_{\min}}\right]$$

Rearranging and solving for the max forward cg position ($h_{\min}$), we get

$$\boxed{\begin{aligned} h_{\min} &= h_n - \frac{\Delta}{a\, C_{L_{\max}}}\left[\frac{C_{m_{0L}} C_{L_\alpha}}{\Delta} + \delta_{e_{\min}}\right] \\ &= h_n - \frac{\Delta}{a\, C_{L_{\max}}}\left[\frac{C_{m_{0L}} C_{L_\alpha}}{\Delta} - \left|\delta_e\right|_{\max\, up}\right] \end{aligned} \qquad (13)}$$