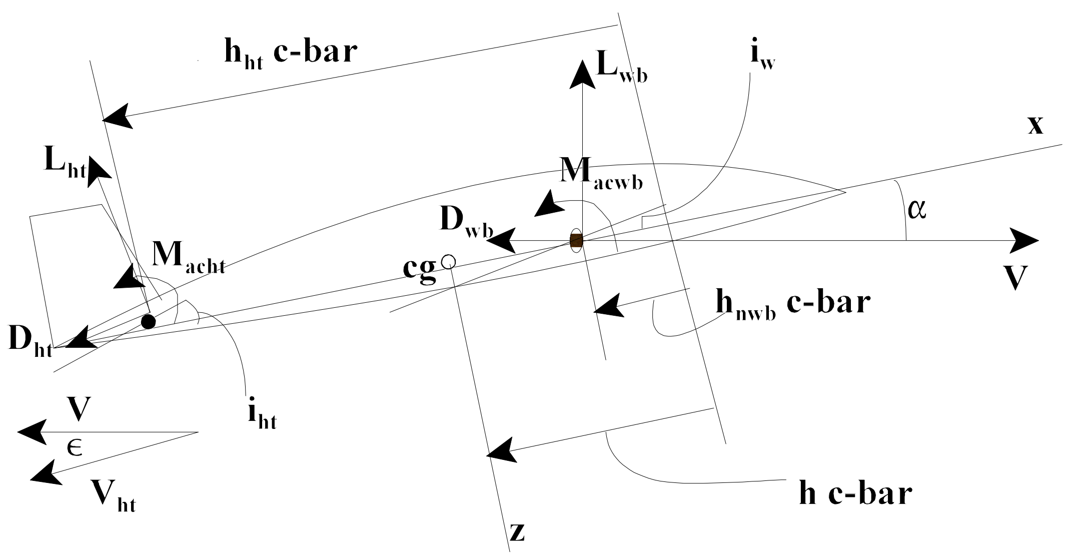
AOE 3134 Complete Aircraft Equations

The requirements for balance and stability that we found for the "flying wing" carry over directly to a complete aircraft. In particular we require the zero-lift pitch moment coefficient of the complete vehicle to be positive and the slope of the pitch-moment curve of the complete vehicle to be negative. Likewise, we require the lift coefficient of the complete vehicle to be sufficient to balance the weight of the vehicle. Our problem then becomes that of determining these quantities for the complete aircraft. In this section we will look at summing the forces and the moments to determine expressions for the complete vehicle lift and pitch moment coefficients.

In order to do these calculations, we can represent the wing and the horizontal tail by their mean aerodynamic chords. The locations of the center-of-gravity, the aerodynamic center of the wing, and the aerodynamic center of the tail are measured from the leading edge of the wing mean aerodynamic chord. We use the mean aerodynamic centers of the wing and horizontal tail as the reference points for applying the lift and drag forces because by definition the pitch-moment about the aerodynamic center does not change with angle-of-attack, and therefor is a constant. The basic layout is given in the figure:



The primary force generators on the vehicle are the lifting surfaces such as the wing and the horizontal tail. In addition, the fuselage can contribute to the lift. Since the contribution of the fuselage to the lift is small compared to that of the wing and the tail, it is convenient to consider the lift of the wing-body combination as is indicated in the figure.

Aircraft Lift Equation

We can sum the forces perpendicular to the relative wind vector in order to determine the vehicle lift. In doing so we can note that the lift of the horizontal tail is perpendicular to the relative wind at the horizontal so that it is "tilted" backward by the downwash angle ϵ. As a result, the sum of the forces perpendicular to the free stream velocity is:

$$L = L_{wb} + L_{ht}\cos\epsilon - D_{ht}\sin\epsilon .$$

If we note that the downwash angle ϵ is small, then we can make the usual small angle approximations: $\cos\epsilon \approx 1, \quad \sin\epsilon \approx \epsilon .$ If, in addition, we make the observation that for a reasonably designed lifting surface (such as the horizontal tail) the drag will be much less than the lift (L/D $\approx$10) then the last term in the previous equation ($D_{ht} * \epsilon$) will be negligible compared to the L_{ht} term. The result of these assumptions is that the total lift on the aircraft can be approximated by the relation:

$$L = L_{wb} + L_{ht},$$

where it should be noted this expression means that the total lift is the sum of the wing-body lift and the tail lift where the wing-body lift is evaluated in the presence of the tail, and the tail lift is evaluated in the presence of the wing-body. For initial design considerations and to discover what properties are important when considering such things as lift, we will see how we can approximate evaluating the lift equation. In general we will include the effect of the wing on the tail lift and ignore the effect of the tail on the wing lift.

The lift the horizontal tail is given by:

$$L_{ht} = C_{L_{ht}}\bar{q}_{ht}S_{ht},$$

where $\bar{q}_{ht}$ is the (average) dynamic pressure at the horizontal tail, and S_{ht} is the horizontal tail area. The lift coefficient of the tail, $C_{L_{ht}}$, is based on the tail area as the reference area. The lift equation becomes:

$$L = L_{wb} + C_{L_{ht}}\bar{q}_{ht}S_{ht}.$$

In order to convert to coefficient form, we can divide the previous equation by the aircraft dynamic pressure, $\bar{q}$, and reference area, S. The result will be an expression for the complete aircraft lift coefficient:

$$\boxed{C_L = C_{L_{wb}} + C_{L_{ht}}\eta_{ht}\frac{S_{ht}}{S}, \qquad (1)}$$

where $\eta_{ht} = \dfrac{\bar{q}_{ht}}{\bar{q}}$ is the ratio of the tail dynamic pressure over the aircraft dynamic pressure and is often called the horizontal tail efficiency. However its value can be greater than, less than, or equal to 1. If in the wake of the wing, the dynamic pressure is generally less than one, if in the wake of a propeller, the value can be greater than one. The second term in Eq. (1) is the lift coefficient of the tail referenced to the aircraft dynamic pressure and reference area. We could write it in the following manner:

$$\left(C_L\right)_{ht} = C_{L_{ht}} \eta_{ht} \frac{S_{ht}}{S},$$

where $\left(C_L\right)_{ht}$ is the contribution to the aircraft lift coefficient due to the horizontal tail.

From the basic ideas from aerodynamics, we can write the lift coefficient for the wing and the tail in the following forms:

$$\begin{aligned} C_{L_{wb}} &= a_{wb}(\alpha_{wb} - \alpha_{0L_{wb}}), \\ C_{L_{ht}} &= a_{ht}(\alpha_{ht} - \alpha_{0L_{ht}}). \end{aligned} \tag{2}$$

However, from the figure we can develop expressions for the local angles-of-attack,

$$\begin{aligned} \alpha_{wb} &= \alpha + i_w, \\ \alpha_{ht} &= \alpha + i_{ht} - \epsilon. \end{aligned} \tag{3}$$

Furthermore, the downwash angle is generally not constant with angle-of-attack and can be modeled as to vary linearly with angle-of-attack in the following way:

$$\begin{aligned} \epsilon &= \epsilon_{0L_{wb}} + \frac{\partial \epsilon}{\partial \alpha}(\alpha_{wb} - \alpha_{0L_{wb}}), \\ &= \epsilon_{0L_{wb}} + \frac{\partial \epsilon}{\partial \alpha}(\alpha + i_w - \alpha_{0L_{wb}}), \\ &= \epsilon_0 + \frac{\partial \epsilon}{\partial \alpha}\alpha, \end{aligned} \tag{4}$$

where ϵ_0 is the downwash at the horizontal tail due to the wing when the angle-of-attack of the aircraft is zero, and $\epsilon_{0L_{wb}}$ is the downwash at the tail when the wing-body lift is zero. Note that in general the two values are different. The relation between the two is obtained from Eq. (4) and is given by:

$$\epsilon_0 = \epsilon_{0L_{wb}} + \frac{\partial \epsilon}{\partial \alpha}(i_w - \alpha_{0L_{wb}}). \tag{5}$$

If we substitute Eq. (4) into Eq. (3), and Eq. (3) into Eq. (2), and Eq. (2) into Eq. (1) we will end up with an expression for the aircraft lift coefficient in terms of the angle-of-attack and other parameters. Some intermediate results of carrying out this activity are:

$$\begin{aligned} C_{L_{ht}} &= a_{ht}\left[\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\alpha + i_{ht} - \epsilon_{0L_{wb}} - \alpha_{0L_{ht}} - \frac{\partial \epsilon}{\partial \alpha}(i_w - \alpha_{0L_{wb}})\right], \\ &= a_{ht}\left[\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\alpha + i_{ht} - \epsilon_0 - \alpha_{0L_{ht}}\right]. \end{aligned} \tag{6}$$

If we complete the exercise, we can write the lift coefficient as a linear term in the angle-of-attack (α) and other constant terms. The resulting expression is the same as that for a lifting surface alone, except now there are no subscripts indicating wing or horizontal tail.

$$\begin{aligned} C_L &= C_{L_0} + C_{L_\alpha}\alpha, \\ &= C_{L_\alpha}(\alpha - \alpha_{0L}) = C_{L_\alpha}\bar{\alpha}, \end{aligned} \tag{7}$$

where C_{L_0} is the aircraft lift coefficient when the angle-of-attack is zero, and α_{0L} is the angle-of-attack when the aircraft lift is zero.

Bt comparing Eq. (7) with the results of our previous substitutions we can observe the following relations:

Aircraft Lift Curve Slope

$$\boxed{C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right].} \tag{8}$$

Aircraft Lift Coefficient at Zero Angle-of-Attack

$$\boxed{\begin{aligned} C_{L_0} &= a_{wb}\left\{(i_w - \alpha_{0L_{wb}}) + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left[i_{ht} - \epsilon_{0L_{wb}} - \alpha_{0L_{ht}} - \frac{\partial \epsilon}{\partial \alpha}(i_w - \alpha_{0L_{wb}})\right]\right\}, \\ &= a_{wb}\left\{(i_w - \alpha_{0L_{wb}}) + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left[i_{ht} - \epsilon_0 - \alpha_{0L_{ht}}\right]\right\}. \end{aligned}} \tag{9}$$

Aircraft Angle-of-Attack at Zero Lift

$$\alpha_{0L} = -\frac{C_{L_0}}{a} = -\frac{a_{wb}}{a}\left\{(i_w - \alpha_{0L_{wb}}) + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left[i_{ht} - \epsilon_{0L_{wb}} - \alpha_{0L_{ht}} - \frac{\partial\epsilon}{\partial\alpha}(i_w - \alpha_{0L_{wb}})\right]\right\},$$
$$= -\frac{a_{wb}}{a}\left\{(i_w - \alpha_{0L_{wb}}) + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left[i_{ht} - \epsilon_0 - \alpha_{0L_{ht}}\right]\right\}. \quad (10)$$

Longitudinal Control

The purpose of the longitudinal control is primarily to provide a pitch-moment increment to the aircraft. A side effect of this increment in pitch-moment is an increment in lift. The primary device for providing longitudinal control is the elevator. The elevator is a flap-like surface attached to the trailing edge of the horizontal tail. This flap can have a chord that is anywhere between 0 and 100 % of the horizontal tail chord. If 100%, the horizontal tail is said to be a “full flying” tail. That is, the whole tail surface deflects. Under these circumstances, the incidence angle of the tail can be considered to be the control. In general, however, the elevator chord is only a fraction of the total horizontal tail chord.

We can define an elevator “lift-curve” slope by considering a tail section to be held at fixed angle-of-attack. Then we can measure the tail lift as the elevator is deflected, and plot the change in lift vs the change in elevator deflection (holding the angle-of-attack constant). The result can be expressed as

$$\Delta C_{L_{ht}} = a_e \delta_e,$$

where a_e is the elevator lift curve slope $a_e = \frac{\Delta C_{L_{ht}}}{\Delta \delta_e} = \frac{\partial C_{L_{ht}}}{\partial \delta_e}$. We can now find the change in the lift coefficient of the complete aircraft by changing the reference velocity and area to that of the aircraft. Consequently, the change in the aircraft lift coefficient due to elevator deflection is given by:

$$\Delta C_L = a_e \eta_{ht} \frac{S_{ht}}{S} \Delta \delta_e.$$

From this equation we can define the ***elevator effectiveness*** as the change in aircraft lift coefficient with elevator deflection as:

$$C_{L_{\delta_e}} = a_e \eta_{ht} \frac{S_{ht}}{S}. \quad (11)$$

We can introduce an alternative way of representing the elevator lift curve by introducing the ratio of the elevator lift curve slope to the tail lift curve slope:

$$\tau_e = \frac{a_e}{a_{ht}} \qquad \Rightarrow \qquad a_e = a_{ht}\,\tau_e,$$

Then the elevator effectiveness can be written as:

$$\boxed{C_{L_{\delta_e}} = \frac{\partial C_L}{\partial \delta_e} = a_{ht}\,\tau_e\,\eta_{ht}\,\frac{S_{ht}}{S} = a_e\,\eta_{ht}\,\frac{S_{ht}}{S}. \qquad (12)}$$

The total contribution to the aircraft lift, (assuming no contribution when δ_e=0) is given by

$$\Delta C_L = C_{L_{\delta_e}}\,\delta_e.$$

We can add this expression to the aircraft lift equation to get the complete aircraft lift equation, including the effect of the longitudinal control (elevator).

The Complete Aircraft Lift Equation

$$\boxed{\begin{aligned} C_L &= C_{L_\alpha}\,\bar{\alpha} + C_{L_{\delta_e}}\,\delta_e, \\ &= C_{L_0} + C_{L_\alpha}\,\alpha + C_{L_{\delta_e}}\,\delta_e, \end{aligned} \qquad (13)}$$

where the ingredients of each term has been presented previously.

Aircraft Moment Equation

The aircraft moment equation is obtained by returning to the figure and writing down the pitch moment about the center-of-gravity. It is convenient to resolve forces along and perpendicular to the reference line (x^b axis). Recall that the distance back to the various points of interest on the aircraft is measured aft from the leading edge of the wing mean aerodynamic chord. We have the following distances of interest:

center-of-gravity location	$= h\,\bar{c}$
wing-body aerodynamic center	$= h_{n_{wb}}\,\bar{c}$
horizontal tail aerodynamic center	$= h_{ht}\,\bar{c}$

The general moment equation is given by summing the moments about the cg. Recall that the velocity at the horizontal tail is, in general, a different magnitude and angle than that of the aircraft and the tail lift and drag are tilted back by an additional amount ϵ, the downwash at the horizontal tail. Summing the moments we have:

$$\begin{aligned} M &= M_{wb} + M_{ht} \\ &= M_{ac_{wb}} + \left(L_{wb}\cos\alpha + D_{wb}\sin\alpha\right)\left(h - h_{n_{wb}}\right)\bar{c} \\ &\qquad + M_{ac_{ht}} - \left[L_{ht}\cos(\alpha - \epsilon) + D_{ht}\sin(\alpha - \epsilon)\right]\left(h_{ht} - h\right)\bar{c} \end{aligned} \tag{14}$$

Although not necessary, we will make a few assumptions to simplify this equation. The usual small angle assumptions, α, and $\alpha - \epsilon$ are small, and their cosines =1.0 and sines = the angle. Further, the drag terms will be assumed small relative to the corresponding lift terms so that the product of the drag term with its corresponding (small) angle is negligible compared to the lift. These are the same assumptions as were used in developing the lift equation. Further, we will assume that the pure horizontal tail pitch moment, $M_{ac_{ht}}$ is negligible compared to the pure wing-body pitch moment, $M_{ac_{wb}}$. If we now divide Eq. (14) by the aircraft dynamic pressure, reference area, and mean aerodynamic chord ($\bar{q}S\bar{c}$), we can convert the equation to coefficient form. In addition we make the same substitutions as we did in the force equation. The result is:

$$\boxed{C_m = C_{m_{0L_{wb}}} + C_{L_{wb}}(h - h_{wb}) - C_{L_{ht}}\eta_{ht}\frac{S_{ht}}{S}(h_{ht} - h)} \tag{15}$$

For balanced flight C_m = 0. Note that this equation, along with Eq. (1) can be used to determine the force distribution between the wing and the horizontal tail. In particular the wing portion of the load is given by $C_{L_{wb}}$, and the horizontal tail portion by $C_{L_{ht}}\eta_{ht}\frac{S_{ht}}{S}$. The two equations, Eq. (1) and Eq. (15) can be used to solve for the two unknowns given the lift coefficient of the aircraft C_L.

The following definitions are useful:

Horizontal Tail Volume

$$\forall_{ht} = \frac{S_{ht}}{S}(h_{ht} - h) = \frac{S_{ht}}{S}\frac{l_{ht}}{\bar{c}} \tag{16}$$

where l_{ht} is defined as the horizontal tail length and is the distance between the center-of-gravity and the aerodynamic center of the horizontal tail. Note that the horizontal tail volume changes if the cg. position is changed.

Modified Tail Volume

$$\bar{\forall}_{ht} = \frac{S_{ht}}{S}(h_{ht} - h_{n_{wb}}) = \frac{S_{ht}}{S}\frac{\bar{l}_{ht}}{\bar{c}},$$

where $\bar{l}_{ht}$ is the modified tail length and is measured from the aerodynamic center of the wing-body to the aerodynamic center of the horizontal tail. Note that here the modified tail volume is a constant, independent of the cg. position.

The two tail volumes can be related by eliminating (h_{ht}) between the two equations. The result is:

$$\forall_{ht} = \bar{\forall}_{ht} - \frac{S_{ht}}{S}(h - h_{n_{wb}}) \tag{17}$$

With these definitions, the pitch moment equation (15) becomes:

$$\boxed{C_m = C_{m_{0L_{wb}}} + C_{L_{wb}}(h - h_{wb}) - C_{L_{ht}}\eta_{ht}\forall_{ht}} \tag{18}$$

We can now replace $C_{L_{wb}}$ and $C_{L_{ht}}$ with the expressions for Eqs. (2 - 4) as we did with the lift equation. The result will be an equation for C_m that is linear in angle of attack, α, plus some constant terms. The coefficient of α will be C_{m_α}, the longitudinal stability parameter.

The Longitudinal Stability Parameter, C_{m_α}

$$\begin{aligned} C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha} &= a_{wb}\left[\left(h - h_{n_{wb}}\right) - \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}(h_{ht} - h)\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right] \\ &= a_{wb}\left[\left(h - h_{n_{wb}}\right) - \frac{a_{ht}}{a_{wb}}\eta_{ht}\forall_{ht}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right] \end{aligned} \tag{19}$$

In order to get the zero lift pitch moment, we can take the lift coefficient equation, Eq. (1) and set $C_L = 0$, or equivalently, set $\alpha = \alpha_{0L}$, as defined in Eq. (10). The result is the pitch moment at zero lift.

The Zero-Lift Pitch Moment Coefficient

$$\begin{aligned} C_{m_{0L}} &= C_{m_{ac_{wb}}} + a_{ht}\,\eta_{ht}\frac{S_{ht}}{S}\left(h_{ht} - h_{wb}\right)\frac{a_{wb}}{a}\left(i_w - \alpha_{0L_{wb}} - i_t + \epsilon_{0_{wb}} + \alpha_{0L_{ht}}\right) \\ &= C_{m_{ac_{wb}}} + a_{ht}\,\eta_{ht}\,\bar{\forall}_{ht}\frac{a_{wb}}{a}\left(i_w - \alpha_{0L_{wb}} - i_t + \epsilon_{0_{wb}} + \alpha_{0L_{ht}}\right) \end{aligned} \tag{20}$$

Pitch Control

The increment in the pitch moment due to elevator deflection can be obtained in the same way we established the increment in the lift coefficient due to elevator control. Starting from basics we have the increment in pitch moment:

$$\begin{aligned} \Delta M &= -l_{ht}\Delta L_{ht} = -(h_{ht} - h)\,\bar{c}\,\Delta L_{ht} \\ &= -(h_{ht} - h)\,\bar{c}\,\Delta C_{L_{ht}}\frac{1}{2}\rho V_{ht}^2 S_{ht} \end{aligned}$$

If we divide by $\bar{q}\,S\,\bar{c}$ we arrive at the increment in pitch moment coefficient due to the elevator deflection. First recall that $\Delta C_{L_{ht}} = a_e\,\delta_e$, then:

$$\begin{aligned} \Delta C_m &= -(h_{ht} - h)\,\eta_{ht}\frac{S_{ht}}{S}\Delta C_{L_{ht}} \\ &= -a_e\,\eta_{ht}\frac{S_{ht}}{S}(h_{ht} - h)\,\delta_e \\ &= -a_e\,\eta_{ht}\,\forall_{ht}\,\delta_e \end{aligned} \tag{21}$$

The coefficient of the elevator deflection is called the ***elevator power***

Elevator Power

$$C_{m_{\delta_e}} = \frac{\partial C_m}{\partial \alpha} = -a_e\,\eta_{ht}\,\forall_{ht} \tag{22}$$

The Aircraft Pitch Moment Equation

Regrouping all the terms, we can rewrite the total aircraft pitch-moment equation including the effects of the elevator control. The total aircraft pitch-moment equation is given by:

$$\boxed{C_m = C_{m_{0L}} + C_{m_\alpha} \bar{\alpha} + C_{m_{\delta_e}} \delta_e \qquad (23)}$$

where all the terms have been developed previously.

Summary

We can gather together all the equations for the aircraft in one spot. The force (lift) and the moment equations for the aircraft are given by:

$$\begin{aligned} C_L &= C_{L_\alpha} \bar{\alpha} + C_{L_{\delta_e}} \delta_e \\ C_m &= C_{m_{0L}} + C_{m_\alpha} \bar{\alpha} + C_{m_{\delta_e}} \delta_e \end{aligned} \qquad (24)$$

where $\bar{\alpha} = \alpha - \alpha_{0L}$, α = angle-of-attack of aircraft, α_{0L} = zero-lift angle of attack of aircraft and $(\cdot)_{0L}$ indicates the value evaluated when $\bar{\alpha}$ is zero and the elevator deflection, δ_e is zero. .

An alternative formulation that uses the aircraft angle-of-attack equal to zero as a reference point can be written as:

$$\begin{aligned} C_L &= C_{L_0} + C_{L_\alpha} \alpha + C_{L_{\delta_e}} \delta_e \\ C_m &= C_{m_0} + C_{m_\alpha} \alpha + C_{m_{\delta_e}} \delta_e \end{aligned} \qquad (25)$$

where α is the aircraft angle-of-attack and $(\cdot)_0$ is the value evaluated when the angle-of-attack, α, is zero and the elevator angle, δ_e.

The terms in these equations are given by:

The zero lift pitch moment:

$$C_{m_{0L}} = C_{m_{ac_{wb}}} + a_{ht} \eta_{ht} \frac{S_{ht}}{S} \left(h_{ht} - h_{n_{wb}}\right) \frac{a_{wb}}{a} \left(i_w - \alpha_{0L_{wb}} - i_{ht} + \epsilon_{0_{wb}} + \alpha_{0L_{ht}} \right) \qquad (26)$$

Lift and Pitch Curve Slope:

$$C_{L_\alpha} = a_{wb}\left[1 + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right]$$
$$C_{m_\alpha} = a_{wb}\left[(h - h_{n_{wb}}) - \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(h_{ht} - h\right)\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right] \quad (27)$$

where:

h = the non-dimensional distance of the cg behind the leading edge of the wing mean aerodynamic chord,

h_{ht} = the non-dimensional distance of the aerodynamic center of the tail behind the leading edge of the wing mean aerodynamic chord.

$h_{n_{wb}}$ = the non-dimensional distance of the aerodynamic center of the wing behind the leading edge of the wing mean aerodynamic chord.

Control Effectiveness and Control Power

$$C_{L_{\delta_e}} = a_e\,\eta_{ht}\frac{S_{ht}}{S}$$
$$C_{m_{\delta_e}} = -a_e\,\eta_{ht}\frac{S_{ht}}{S}\left(h_{ht} - h\right) = -a_e\,\eta_{ht}\forall_{ht} \quad (28)$$

where a_e = the "elevator lift-curve slope," or the change in horizontal tail lift when the elevator is deflected and the horizontal tail is held at constant angle-of-attack

$\forall_{ht}$ = horizontal tail volume

For the alternative formulation we need the additional relation (obtained by setting $\alpha = 0$),

Lift and Pitch-Moment at Zero Angle-of-Attack

$$C_{L_0} = a_{wb}\left[(i_w - \alpha_{0L_{wb}}) + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(i_{ht} - \epsilon_0 - \alpha_{0L_{ht}}\right)\right]$$
$$C_{m_0} = C_{m_{ac_{wb}}} + a_{wb}(i_w - \alpha_{0L_{wb}})(h - h_{n_{wb}}) - a_{ht}(i_{ht} - \epsilon_0 - \alpha_{0L_{ht}})\frac{S_{ht}}{S}\left(h_{ht} - h\right)\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) \quad (29)$$