AOE 3134 Stability and Control
Important Equations and Relations

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1. Longitudinal Static Stability

a. General

i. General moment relation between two reference points

$$C_{m_{(b)}} = C_{m_{(a)}} + C_L\,(h_b - h_a)$$

ii. Aerodynamic center: $C_{m_{AC}}$ = constant independent of angle-of-attack

$$C_{m_{AC}} = C_{m_{0L}}$$

$$C_{m_{(x)}} = C_{m_{0L}} + C_L\,(h_{(x)} - h_{ac}) = C_{m_{0L}} + C_L\,(h_{(x)} - h_n)$$

b. Lift related characteristics

i. Lift curve slope of aircraft $C_{L_\alpha} = a$

$$C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right]$$

ii. Lift coefficient at zero angle-of-attack C_{L_0}

(Note: usually $i_w = 0$, it is not zero if the wing is treated separately from the fuselage, then $a_{wb} = a_w$, etc.)

$$C_{L_0} = a_{wb}\left\{(i_w - \alpha_{0L_{wb}}) + \frac{a_{ht}}{a_{wb}}\eta_{ht}\frac{S_{ht}}{S}\left[i_{ht} - \epsilon_{0_{wb}} - \alpha_{0L_{ht}} - \frac{\partial\epsilon}{\partial\alpha}(i_w - \alpha_{0L_{wb}})\right]\right\}$$

iii. Aircraft zero-lift angle of attack, α_{0L}

$$\alpha_{0L} = -\frac{C_{L_0}}{a}$$

iv. Elevator control deflection contribution to lift, $C_{L_{\delta_e}}$

$$C_{L_{\delta_e}} = a_e \eta_{ht} \frac{S_{ht}}{S} = a_{ht} \tau_e \eta_{ht} \frac{S_{ht}}{S}$$

Note: For full "flying tail" $\tau_e = 1$, and $\delta_e = \Delta i_{ht}$

v. Total tail lift coefficient

$$C_{L_{ht}} = a_{ht} \bar{\alpha}_{ht} + a_e \delta_e$$

vi. Aircraft lift characteristics

$$C_L = C_{L_\alpha} (\alpha - \alpha_{0L}) + C_{L_{\delta_e}} \delta_e$$

$$C_L = C_{L_\alpha} \bar{\alpha} + C_{L_{\delta_e}} \delta_e = a \bar{\alpha} + C_{L_{\delta_e}} \delta_e$$

$$C_L = C_{L_0} + C_{L_\alpha} \alpha + C_{L_{\delta_e}} \delta_e$$

Also

$$C_L = C_{L_{wb}} + C_{L_{ht}} \eta_{ht} \frac{S_{ht}}{S}$$

c. Moment related characteristics

i. Aircraft moment

$$C_m = C_{m_{0L_{wb}}} + C_{L_{wb}} (h - h_{n_{wb}}) - \eta_{ht} V_{ht} C_{L_{ht}}$$

$$C_m = C_{m_{0L_w}} + C_{L_w} (h - h_{n_w}) - \eta_{ht} V_{ht} C_{L_{ht}} + C_{m_{fn}}$$

where

$$V_{ht} = \frac{S_{ht}}{S} \frac{l_{ht}}{\bar{c}} = \frac{S_{ht}}{S} \cdot (h_{ht} - h)$$

$$l_{ht} = (h_{ht} - h) \cdot \bar{c} \doteq \text{tail length}$$

also $$\bar{l}_{ht} = (h_{ht} - h_{n_{wb}})\,\bar{c} \doteq \text{modified tail length}$$

ii. Aircraft longitudinal stability parameter

$$C_{m_\alpha} = a_{wb}\left[(h - h_{n_{wb}}) - \frac{a_{ht}}{a_{wb}}\eta_{ht}V_{ht}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right]$$

$$C_{m_\alpha} = a_{w}\left[(h - h_{n_{w}}) - \frac{a_{ht}}{a_{w}}\eta_{ht}V_{ht}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right] + \left.\frac{\partial C_m}{\partial\alpha}\right)_{fn}$$

The wing-fuselage aerodynamic center location is related to the wing aerodynamic center location by the following approximate relationship:

$$h_{n_{wb}} = h_{n_w} - \frac{1}{a_w}\left.\frac{\partial C_m}{\partial\alpha}\right)_{fn}$$

iii. Aircraft zero lift moment

$$C_{m_{0L}} = C_{m_{0L_{wb}}} + a_{ht}\eta_{ht}\frac{S_{ht}}{S}(h_{ht} - h_{n_{wb}})\frac{a_{wb}}{a}(i_w - \alpha_{0L_{wb}} - i_{ht} + \epsilon_{0L_{wb}} + \alpha_{0L_{ht}})$$

iv. Aircraft pitch control power

$$C_{m_{\delta_e}} = -a_e\eta_{ht}V_{ht}$$

v. Aircraft pitch-moment equation

$$C_m = C_{m_{0L}} + C_{m_\alpha}\bar{\alpha} + C_{m_{\delta_e}}\delta_e$$

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta_e}}\delta_e$$

d. Stick fixed stability parameters

i. Stick fixed neutral point (Aircraft aerodynamic center)

$$C_{m_\alpha} = a(h - h_n)$$

where

$$h_n = \frac{h_{n_{wb}} + \frac{a_{ht}}{a_{wb}} \eta_{ht} \frac{S_{ht}}{S} h_{ht} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)}{1 + \frac{a_{ht}}{a_{wb}} \eta_{ht} \frac{S_{ht}}{S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)}$$

$$h_n = h_{n_{wb}} + \frac{a_{ht}}{a} \eta_{ht} \frac{S_{ht}}{S} (h_{ht} - h_{n_{wb}}) \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)$$

ii. Stick fixed static margin

$$K_n \doteq h_n - h$$

iii. Maximum forward cg position (from balanced-level-flight-control point of view)

$$h_{\min} = h_n - \frac{\Delta}{a\, C_{L_{\max}}} \left[\frac{C_{m_{0L}} C_{L_\alpha}}{\Delta} - | \delta_e |_{\substack{\max \\ up}} \right]$$

2. Aircraft equilibrium (balanced flight) equations

a. Balanced flight equations

$$C_L = C_{L_\alpha} \bar{\alpha} + C_{L_{\delta_e}} \delta_e \qquad = \frac{L}{1/2\, \rho\, V^2 S}$$

$$C_m = C_{m_{0L}} + C_{m_\alpha} \bar{\alpha} + C_{m_{\delta_e}} \delta_e = 0$$

b. Elevator angle and angle of attack for balanced flight

$$\bar{\alpha} = \frac{C_{L_{\delta_e}} C_{m_{0L}} + C_{m_{\delta_e}} C_L}{\Delta}$$

$$\delta_e = - \left[\frac{C_{L_\alpha} C_{m_{0L}} + C_{m_\alpha} C_L}{\Delta} \right]$$

where

$$\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{m_\alpha} C_{L_{\delta_e}}$$

3. Longitudinal control surface considerations

a. Elevator hinge moment

$$C_{H_e} = \frac{H_e}{\frac{1}{2} \rho V_{ht}^2 S_e \bar{c}_e}$$

$$= C_{h_{e_0}} + \frac{\partial C_{H_e}}{\partial \alpha_{ht}} \bar{\alpha}_{ht} + \frac{\partial C_{H_e}}{\partial \delta_e} \delta_e + \frac{\partial C_{H_e}}{\partial \delta_t} \delta_t$$

$$= b_0 + b_1 \bar{\alpha}_{ht} + b_2 \delta_e + b_3 \delta_t$$

b. Stick free, $C_{H_e} = 0$ (assume $b_0 = 0$ and $\delta_t = 0$, for convenience only)

i. Elevator float angle

$$\delta_e = -\frac{b_1}{b_2} \alpha_{ht}$$

c. Stick free properties - replace a_{ht} with $\hat{a}_{ht} = F a_{ht}$

where $F = 1 - \frac{a_e}{a_{ht}} \frac{b_1}{b_2}$ = the free elevator factor

i. Lift-curve slope

$$\hat{a} = a_{wb} \left[1 + F \frac{a_{ht}}{a_{wb}} \eta_{ht} \frac{S_{ht}}{S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

ii. Longitudinal stability parameter

$$\hat{C}_{m_\alpha} = a_{wb}\left[h - h_{n_{wb}} - F \frac{a_{ht}}{a_{wb}} \eta_{ht} V_{ht} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

iii. Stick free neutral point

$$\hat{h}_n = \frac{h_{n_{wb}} + F \frac{a_{ht}}{a_{wb}} \eta_{ht} \frac{S_{ht}}{S} h_{ht} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)}{1 + F \frac{a_{ht}}{a_{wb}} \eta_{ht} \frac{S_{ht}}{S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)}$$

$$\hat{h}_n = h_{n_{wb}} + F \frac{a_{ht}}{\hat{a}} \eta_{ht} \frac{S_{ht}}{S} (h_{ht} - h_{n_{wb}}) \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)$$

4. Stick force characteristics

a. Stick force - general

$$F_s = G \, C_{H_e} \eta_{ht} \frac{1}{2} \rho V^2 S_e \bar{c}_e$$

b. Stick force at off-trimmed conditions

$$F_s = G \eta_{ht} S_e \bar{c}_e \frac{W}{S} \frac{b_2}{\Delta} \hat{a} (h - \hat{h}_n) \left(\frac{V^2}{V_{trim}^2} - 1 \right)$$

c. stick-force gradient

$$\left. \frac{d F_s}{d V} \right|_{V_{trim}} = 2 G \eta_{ht} S_e \bar{c}_e \frac{W}{S} \frac{b_2}{\Delta} \hat{a} \frac{(h - \hat{h}_n)}{V_{trim}}$$

5. Maneuvering Flight

a. Pitch damping stability derivatives

$$C_{m_q} = \frac{\partial C_m}{\partial\left(\frac{q\,\bar{c}}{2\,V}\right)} = -2\,K\,a_{ht}\sqrt{(\eta_{ht})}\,V_{ht}\,\frac{l_{ht}}{\bar{c}}$$

where K = 1.1

$$C_{L_q} = \frac{\partial C_L}{\partial\left(\frac{q\,\bar{c}}{2\,V}\right)} = 2\,a_{ht}\sqrt{\eta_{ht}}\,V_{ht}$$

b. Elevator angle requirements , (Fac) = 1 for pull-up, = (n+1)/n for horizontal turn

i. Elevator increment for maneuver

$$\Delta\delta_e = -\frac{(n-1)\,C_w\left[C_{m_\alpha} + (C_{L_\alpha}C_{m_q} - C_{m_\alpha}C_{L_q})\,(Fac)\,\frac{\rho\,S\,\bar{c}}{4\,m}\right]}{\Delta}$$

ii. Elevator angle / g

$$\frac{\partial\,\Delta\,\delta_e}{\partial\,n} = -\frac{C_w\left[C_{m_\alpha} + (C_{L_\alpha}C_{m_q} - C_{m_\alpha}C_{L_q})\,(Fac)\,\frac{\rho\,S\,\bar{c}}{4\,m}\right]}{\Delta}$$

$$= -\frac{C_w\,C_{L_\alpha}\left(\frac{4\,m}{\rho\,S\,\bar{c}} - C_{L_q}\right)}{\frac{4\,m}{\rho\,S\,\bar{c}}\,\Delta}\,(h - h_{mp}) \qquad (\text{pull up})$$

iii. Stick fixed maneuver point

$$h_{mp} = h_n - \frac{C_{m_q}}{\left(\frac{4\,m}{\rho\,S\,\bar{c}}\right) - C_{L_q}}$$

iv. Stick force / g

$$\frac{\partial(\Delta F_s)}{\partial n} = -G\,\eta_{ht}\,S_e\,\bar{c}_e\,\frac{W}{S}\,\hat{a}\,b_2\,\frac{\left(\frac{4m}{\rho S \bar{c}} - C_{L_q}\right)}{\frac{4m}{\rho S \bar{c}}\,\Delta}\,(h - \hat{h}_{mp})$$

where the stick free maneuver point $\hat{h}_{mp}$ is given by

$$\hat{h}_{mp} = h_{mp} + \frac{\Delta}{\hat{a}\,b_2}\left[\frac{b_1\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)}{a} + \frac{2\,b_1\,(h_{ht} - h)}{\sqrt{\eta_{ht}}\left(\frac{4m}{\rho S \bar{c}} - C_{L_q}\right)}\right]$$

Note: b_1 in the third term on the right must be in units of /rad!

7. Lateral directional characteristics

a. Yawing moment

i. Aircraft yaw moment

$$\begin{aligned} C_n &= C_{n_{wb}} - \eta_{vt}\,V_{vt}\,C_{L_{vt}} \\ &= C_{n_{wb}} - \eta_{vt}\,V_{vt}\,[a_{vt}(\sigma - \beta) + a_r\,\delta_r] \\ &= C_{n_\beta}\,\beta + C_{n_{\delta_r}}\,\delta_r \end{aligned}$$

ii. Directional stability parameter C_{n_β}

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta} = C_{n_\beta}\Big|_{wb} + a_{vt}\,\eta_{vt}\,V_{vt}\left(1 - \frac{\partial \sigma}{\partial \beta}\right) > 0 \quad \text{for stability}$$

iii. Directional (rudder) control power

$$C_{n_{\delta_r}} = \frac{\partial C_n}{\partial \delta_r} = -a_r\,\eta_{vt}\,V_{vt} = -a_{vt}\,\tau_r\,\eta_{vt}\,V_{vt}$$

iv. Stick free stability parameter

$$\hat{C}_{n_\beta} = C_{n_\beta}\Big|_{wb} + F_r\, a_{vt}\, \eta_{vt}\, V_{vt} \left(1 - \frac{\partial \sigma}{\partial \alpha} \right)$$

where

$$F_r = 1 - \frac{a_r}{a_{vt}} \frac{b_{1_{vt}}}{b_{2_{vt}}}$$

v. Stick force and stick force gradient

$$F_{s_r} = -G_r \frac{1}{2} \rho V^2 \eta_{vt} S_r \bar{c}_r \left[\left(b_{1_r} + b_{2_r} \frac{C_{n_\beta}}{C_{n_{\delta_r}}} \right) \beta - b_1 \sigma \right]$$

$$\frac{d F_{s_r}}{d \beta} = - \frac{G \frac{1}{2} \rho V^2 \eta_{vt} S_r \bar{c}_r b_{r_r}}{C_{n_{\delta_r}}} \hat{C}_{n_\beta}$$

b. Roll stability parameter (vertical tail contribution only)

$$C_{l_\beta} = a_{vt} \eta_{vt} V_{vt} \left(1 - \frac{\partial \sigma}{\partial \beta} \right) \frac{z_{vt}}{l_{vt}} < 0 \text{ for stability}$$

8. General lateral-directional steady state equations

$$\begin{bmatrix} C_{Y_\beta} & C_{Y_{\delta_a}} & C_{Y_{\delta_r}} & C_{Y_p} & C_{Y_r} & C_w \\ C_{l_\beta} & C_{l_{\delta_a}} & C_{l_{\delta_r}} & C_{l_p} & C_{l_r} & 0 \\ C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} & C_{n_p} & C_{n_r} & 0 \end{bmatrix} \begin{Bmatrix} \beta \\ \delta_a \\ \delta_r \\ \hat{p} \\ \hat{r} \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -C_{n_T} \end{Bmatrix}$$

Pick a set of three variables and solve for the remaining three. For a steady turn

$$\hat{p} = -\frac{\omega b}{2V}\sin\theta$$

$$\hat{r} = \frac{\omega b}{2V}\cos\phi\cos\theta$$

9. Some aerodynamic estimation formulas

a. Lift curve slope

$$a_{\substack{wing \\ tail}} = \frac{2\pi AR}{2 + \sqrt{\frac{AR^2(1 - M^2)}{k^2}\left(1 + \frac{\tan^2\Lambda_{c/2}}{(1 - M^2)}\right) + 4}}$$

where $k = \dfrac{c_{l_\alpha}}{2\pi}$, actual 2-D lift-curve slope / theoretical-flat-plate 2-D lift-curve slope

b. Downwash parameter

$$\frac{\partial\epsilon}{\partial\alpha} = 4.44\left[K_a K_\lambda K_H\left(\cos\Lambda_{\frac{c}{4}}\right)^{\frac{1}{2}}\right]^{1.19}$$

where

$$K_A = \frac{1}{AR} - \frac{1}{1 + AR^{1.7}}$$

$$K_\lambda = \frac{10 - 3\lambda}{7}$$

$$K_H = \frac{1 - \left|\frac{h_H}{b}\right|}{\sqrt[3]{\frac{2\bar{l}_{ht}}{b}}} \qquad \text{where} \quad \bar{l}_{ht} = (h_{ht} - h_{n_{wb}})\bar{c}$$

11. Basic Equations of Motion

$\bar{V}^b = u\hat{i}^b + v\hat{j}^b + w\hat{k}^b$ is the velocity vector represented in body fixed coordinates.

$\bar{\omega} = p\hat{i}^b + q\hat{j}^b + r\hat{k}^b$ is the angular rate vector represented in body fixed coordinates.

Then the differential equations of motion are:

a. Force Equations

$$F_x = m(\dot{u} + qw - rv)$$

$$F_y = m(\dot{v} + ru - pw)$$

$$F_z = m(\dot{w} + pv - qu)$$

b. Moment Equations

$$L = I_x\dot{p} + (I_z - I_y)qr - I_{xz}(pq + \dot{r})$$

$$M = I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2)$$

$$N = I_z\dot{r} + (I_y - I_x)pq + I_{xz}(rq - \dot{p})$$

c. Kinematic Equations

$$\dot{\phi} = p + \tan\theta\,(q\sin\phi + r\cos\phi)$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\psi = \sec\theta\,(q\sin\phi + r\cos\phi)$$

2. Forces - Aerodynamic, Gravitational, Thrust

a. Aerodynamic (for body fixed stability axes) and Thrust

$Fx_T + Fx_A = T_x$ - Drag $Fy_T + Fy_A = T_y + Y$ $Fz_T + Fz_A = T_z$ - Lift

b. Gravitational (for body fixed (any) axes)

$Fx_g = -mg\sin\theta$ $Fy_g = mg\cos\theta\sin\phi$ $Fz_g = mg\cos\theta\cos\phi$