Problem Sheet Eight - Answers

30. This problem will compute values needed in problem 31 also.

Sea Level: $M_a = 0.25, \quad a = 1116.4 \text{ ft/sec} \qquad V = M_a a = 0.25\,(1116.4) = 279.10 \text{ ft/sec}$

$\rho = 0.00237 \text{ slug/ft}^3, \qquad qbqr = 1/2\,\rho\,V^2 = 1/2\,(0.002377)\,(279.10)^2 = 92.5804 \text{ lbs/ft}^2$

$L_p = c_{l_p} \dfrac{\bar{q}\,S\,b^2}{2\,V} = -0.45\,\dfrac{92.5804\,(5500)\,(195.68)^2}{2\,(279.10)} = -15718000.68 \text{ ft}-\text{lbs/rad/sec}$

$\dfrac{L_p}{I_x} = L_p/18200000 = -0.8636 \text{ /sec}$

$L_{\delta_a} = C_{l_{\delta_a}}\,\bar{q}\,S\,b = 0.0461\,(92.5804)\,5500\,(195.68) = 4593345.439 \text{ ft lbs/rad}$

$$\Delta\,p(t) = -\frac{L_{\delta_a}\,\delta_a}{L_p}\left[1 - e^{-\frac{L_p}{I_x}t}\right]$$

$$\Delta\,p(t) = -\frac{4593345.439}{-15718000.68}\left(\frac{5\,\pi}{180}\right)\left[1 - e^{-0.8636\,t}\right]$$

$$\underline{\Delta\,p(t) = 0.0255\left(1 - e^{-0.8636\,t}\right)}$$

$$\Delta\,\phi(t) = 0.0255\int_0^t \left(1 - e^{-0.8636\,t}\right)dt = 0.0255\left[t - \frac{e^{-0.8636\,t}}{-0.8636}\right]_0^t$$

$$\underline{\Delta\phi(t) = 0.0255\,t + 0.0295\left[e^{-0.8636\,t} - 1\right]}$$

At 40k ft;

$V = M_a\,a = 0.9\,(968.08) = 871.2720 \text{ ft/sec} \qquad \rho = 0.000587 \text{ slug/ft}^3$

$\bar{q} = 1/2\,\rho\,V^2 = 1/2\,(0.000587)\,871.272^2 = 222.8002 \text{ lb/ft}^2$

$L_p = C_l \dfrac{\bar{q}\,S\,b^2}{2\,V} = -0.30\,\dfrac{(222.8002)115500\,(195.68)^2}{2\,(871.212)} = =8078090.06 \text{ ft lbs/ rad/sec}$

$\dfrac{L_p}{I_x} = \dfrac{-8078090.06}{18200000} = -0.4438 \text{ /sec}$

$L_{\delta_a} = C_{l_{\delta_a}}\,\bar{q}\,S\,b = 0.014\,(222.8002)\,5500\,(195.68) = 3357010.821 \text{ ft lbs/rad}$

Inserting into the equations above for *p(t)* and $\phi(t)$, we get

$$\Delta p(t) = 0.0363\left[1 - e^{-0.4438\,t}\right]$$

and

$$\Delta \phi(t) = 0.0363\,t + 0.0817\left[e^{-0.4438\,t} - 1\right]$$

31. Aileron input at full control = 30 deg

a) Sea level:

$$\Delta p_{ss} - \frac{L_{\delta_a}}{L_p}\delta_a = -\frac{4593345.439}{-15718000.68}\left(\frac{30\,\pi}{180}\right) = 0.1530 \text{ rad/sec} = 8.766 \text{ deg/sec}$$

40k ft

$$\Delta p_{ss} = -\frac{3357010.821}{-8078090.06}\left(\frac{30\,\pi}{180}\right) = 0.2176 \text{ rad/sec} = 12.467 \text{ deg/sec}$$

b) General Solution

$$\Delta p(t) = \Delta p_{ss}\left[1 - e^{a\,t}\right] + \Delta p_0\, e^{a\,t}$$

integrating:

$$\Delta\phi(t) = \Delta p_{ss}\left[t - \frac{1}{a}\left(e^{a\,t} - 1\right)\right] + \frac{\Delta p_0}{a}\left(e^{a\,t} - 1\right)$$

(Note that this version of the general solution is needed if you decide to do the "extra problem." However for the case of interest here, we have $\Delta p_0 = 0$, and can substitute numbers in for the sea level case:

$$\begin{aligned}\Delta \phi(t) &= 0.1530\left[t - \frac{1}{-0.8636}\left(e^{-0.8636\,t} - 1\right)\right]\\ &= 0.1530\left[t + 1.5794\left(e^{-0.8636\,t} - 1\right)\right]\end{aligned}$$

We can now specify an angle, and solve for the time. You can use Mathcad. Matlab, Mathematica, or some other math package to solve these for t, or you can simply use Newton's method for solving nonlinear equations. Newton's method is simply an algorithm that starts with an initial guess t_0 and then tells us how to improve that guess. The general from of this method is:

$$t_{k+1} = t_k - \frac{f(t_k)}{f'(t_k)}$$

where $f(t) = 0$ at the solution.

In our problem

$$f(t) = 0.1530\left[t + 1.5794\left(e^{-0.8636\,t} - 1\right)\right] - \phi_{desired}$$

and

$$f'(t) = \frac{d\,\Delta\phi(t)}{dt} = \Delta p(t) = 0.1530\left(1 - e^{-0.8636\,t}\right)$$

Hence for this problem Newton's method takes the form:

$$t_{k+1} = t_k - \frac{0.1530\left[t_k + 1.5794\left(e^{-0.8636\,t} - 1\right)\right] - \phi_{desired}}{0.153\left(1 - e^{-0.8636\,t}\right)}$$

Iterate until the magnitude of $|\,t_{k+1} - t_k\,|$ is less than some specified tolerance.

The results for various angles are;

$$\begin{aligned} \pi/2 &= 90\ \text{deg} & t &= 11.42\ \text{sec} \\ \pi &= 180\ \text{deg} & t &= 21.69\ \text{sec} \\ 2\,\pi &= 360\ \text{deg} & t &= 42.22\ \text{sec} \end{aligned}$$

32. $\Delta\ddot{\beta} - \dfrac{1}{I_z}(N_r - N_{\dot{\beta}})\,\Delta\dot{\beta} + \dfrac{N_\beta}{I_z}\,\Delta\beta = 0$

$$N_r = C_{n_r}\frac{\bar{q}\,S\,b^2}{2\,V} = -0.30\,\frac{92.5804\,(5500)\,(195.68)^2}{2\,279.10} = -10491887.5\ \text{ ft lbs/rad/sec}$$

$$\frac{N_r}{I_x} = \frac{-10491887.5}{49700000} = -0.3011\ \text{/sec}$$

$$N_\beta = C_{n_\beta}\,\bar{q}\,S\,b = 0.150\,(92.5804)\,5500\,(195.68) = 14964665.8\ \text{ ft lbs/rad}$$

$$\frac{N_\beta}{I_x} = \frac{14964665.8}{49700000} = 0.3011\ \text{/sec}^2$$

Substituting into our basic equation:

$$\Delta\ddot{\beta} + 0.211\,\Delta\dot{\beta} + 0.3011\,\delta\beta = 0$$

The characteristic equation (obtained by assuming a solution $\Delta\beta = C\,e^{\lambda t}$

$$\lambda^2 + 0.211\,\lambda + 0.3011 = 0$$

The solution to the characteristic equation gives us the characteristic values (eigenvalues) of the system. For this problem they are:

$$\begin{aligned}\lambda &= n \pm i\,\omega \\ &= -0.1055 \pm i\,0.5385\end{aligned}$$

We can immediately determine the following properties of the motion.

$$\omega = 0.5385 \text{ rad/sec} \qquad \Rightarrow \qquad T_p = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{0.5385} = 11.67 \text{ sec}$$

Note that this **is not** ω_n. In addition we have

$$\zeta = \frac{|n|}{\omega_n} = \frac{|n|}{\sqrt{n^2 + \omega^2}} = \frac{0.1055}{\sqrt{0.1055^2 + 0.5385^2}} = \frac{0.1055}{0.3011} = 0.1923$$

33. Assumptions: Model has same density as aircraft
Model has same aerodynamic characteristics
Both at same velocity

$$N_r = C_{n_r} \frac{\bar{q}\,S\,b^2}{2\,V} = C_{n_r} \frac{\bar{q}\,(l^2)\,(l^2)}{2\,V} \propto l^4$$

and

$$I_z = m\,l^3 = \rho\,l^3\,l^2 \propto l^5$$

Hence:

$$\frac{N_r}{I_z} \propto \frac{l^4}{l^4} = \frac{1}{l} \qquad \Rightarrow \qquad \left.\frac{N_r}{I_x}\right)_{model} = 10 \left.\frac{N_r}{I_x}\right)_{aircraft} = -2.111 \text{ /sec}$$

Similarly,

$$\frac{N_\beta}{I_x} = C_{n_\beta} \frac{\bar{q}\,(l^2)\,(l)}{2\,V\,\rho\,l^5} \propto \frac{1}{l^2} \qquad \Rightarrow \qquad N_\beta\big)_{model} = 100 \left.\frac{N_\beta}{I_x}\right)_{A/C} = 30.11 \text{ /sec}^2$$

For the model:

$$\Delta\ddot{\beta} + 2.111\,\Delta\dot{\beta} + 30.11\,\Delta\beta = 0$$

The solution to the characteristic equation ($\lambda^2 + 2.111\,\lambda + 30.11 = 0$) is:

$$\lambda = -1.0555 \pm i\,5.3848$$

The corresponding properties:

$\omega = 5.3848$ rad/sec $\qquad T_p = 1.167$ sec $\qquad \zeta - 0.1923$

34. The governing equation of motion for the pitch oscillation about the y axis is

$$\Delta\ddot{\alpha} - \frac{1}{I_y}\left(M_q + M_{\dot{\alpha}}\right)\Delta\,\dot{\alpha} - \frac{1}{I_y} M_\alpha\,\Delta\,\alpha = 0$$

Calculating the ingredients we have:

$$M_q = C_{m_q}\frac{\bar{q}\,S\,\bar{c}}{2\,V} = -12.4\,\frac{49.6\,(174)\,4.9^2}{2\,(220.5)} = -5826.48 \text{ ft lbs/rad/sec}$$

$$\frac{M_q}{I_y} = \frac{-5826.48}{1346} = -4.329 \text{ /sec}$$

$$M_{\dot{\alpha}} = C_{m_{\dot{\alpha}}}\frac{\bar{q}\,S\,\bar{c}^2}{2\,V} = -7.27\,\frac{49.6\,(174)\,(4.9)^2}{2\,(220.5)} = -3416.08 \text{ ft lbs/rad/sec}$$

$$\frac{M_{\dot{\alpha}}}{I_y} = \frac{-3416.08}{1346} = -2.538 \text{ /sec}$$

$$M_\alpha = C_{m_\alpha}\,\bar{q}\,S\,\bar{c} = -0.613\,(49.6)\,(174)\,4.9 = -25923.13 \text{ ft lbs/rad}$$

$$\frac{M_\alpha}{I_y} = \frac{-25923.132}{1346} = -19.259 \text{ /sec}^2$$

Then:

$$\Delta\ddot{\alpha} + (4.329 + 2.538\,)\,\Delta\,\dot{\alpha} + 19.259\,\Delta\,\alpha = 0$$
$$\Delta\,\ddot{\alpha} + 6.867\,\Delta\,\dot{\alpha} + 19.259\,\Delta = 0$$

The characteristic equation is:

$$\lambda^2 + 6.867\,\lambda + 19.259 = 0$$

“The Standard Form”

$$\lambda^2 + 2\,\zeta\,\omega_n\,\lambda + \omega_n^2 = 0$$

The solution:

$$\boxed{\lambda = -3.433 \pm i\,2.733 \text{ /sec}}$$

a) $t_{1/2} = \frac{\ln 2}{|n|} = \frac{\ln 2}{3.433} = 0.202 \text{ sec}$

b) $\omega = 2.733 \text{ rad/sec} \cdot \frac{1 \, cycle}{2 \pi \text{ rad}} = 0.425 \text{ hz}$

c) $\zeta = \frac{2 \zeta \omega_n}{2 \omega_n} = \frac{6.867}{2\sqrt{19.259}} = 0.782$ Or $\zeta = \frac{|n|}{\sqrt{n^2 + \omega^2}} = \frac{3.433}{\sqrt{3.433^2 + 2.733^2}} = 0.782$

d) $T_p = \frac{2\pi}{\omega} = \frac{2\pi}{2.733} = 2.299 \text{ sec}$ $N_{1/2} = \frac{t_{1/2}}{T_p} = \frac{0.202}{2.299} = 0.0878$ cycles to half amplitude

e) Moving the cg forward will make C_{m_α} more negative.

$$C_{m_\alpha} = a(h - h_n)$$

This change, in turn, will make M_α more negative and correspondingly will make the last term in the differential equation larger. Hence ω_n^2 is larger and ω_n is thus larger.

On the other hand, $C_{m_q} = -a_{ht} \eta_{vt} \forall_{ht} \frac{l_{ht}}{\bar{c}} = -a_{ht} \eta_{ht} \frac{S_{ht}}{S} \frac{l_{ht}^2}{\bar{c}^2}$

will change since l_{ht} will increase. However the change in C_{m_q} will be a much smaller percent then that of C_{m_α} because $(h - h_n)$ is smaller to begin with then $(h_{ht} - h)$, so for a small change in h, the percent change in the value is much less. So for starters we will assume that the terms associated with M_q and $M_{\dot{\alpha}}$ do not change significantly. Assuming that is the case, then we can look at the standard form:

$$\ddot{\lambda} + 2 \zeta \omega_n \dot{\lambda} + \omega_n^2 = 0$$

From our assumptions above, the middle term is unchanged, and the last term increases.
The solution is

$\lambda = n \pm i\omega = -\zeta \omega_n \pm i \omega_n \sqrt{1 - \zeta^2}$

The result of these changes are:

since $2 \zeta \omega_n$ is unchanged, the n is unchanged and

a) $t_{1/2} = \frac{\ln 2}{|n|}$ $\Rightarrow$ unchanged

c) $\zeta \omega_n \approx const \quad \Rightarrow \quad \zeta = \frac{1}{\omega_n}$ Since ω_n increases, ζ decreases>

b) $\omega = \omega_n \sqrt{1 - \zeta^2} \quad \Rightarrow$ Since ω_n is larger, and ζ is smaller, ω is larger.

d) $T_p = \frac{2\pi}{\omega}$ Therefore T_p is smaller and $\frac{t_{1/2}}{T_p}$ becomes larger.