6. Changing Orbits

Once an orbit is established in the two body problem, it will remain the same size (semi major axis) and shape (eccentricity) in the original orbit plane. In order to change from one orbit to another, we must either change the velocity vector in either magnitude or direction or both. The assumption that will be made for this introductory study of changing orbits is that we are able to change velocity (magnitude and/or direction) instantaneously, without changing the radius vector. Such an assumption is called the impulsive thrust assumption. If we consider an impulsive thrust per unit mass of the vehicle, this will be equivalent to the change in velocity, what we usually designate as $\Delta\vec{V}$. Consequently, we seek the $\Delta\vec{V}$ required to perform some specific orbit change. This change could be in size, shape, and orbit plane. We should also note that this $\Delta\vec{V}$ is a vector and must be treated as such. Further, we will show later that the fuel burned in order to generate such a $\Delta\vec{V}$ is related to the magnitude of $\Delta\vec{V}$, $|\Delta\vec{V}| = \Delta V$. If we initially assume that our $\Delta\vec{V}$ will be added tangentially (either in the same direction or opposite direction of the current orbital velocity), then we can either add or subtract the magnitude of the $\Delta\vec{V}$ or ΔV to or from the magnitude of the current velocity, and recalculate the new orbit properties. This scenario is the simplest one and will be studied first.

Applying $\Delta\vec{V}$ Tangentially

Since in this section we will be applying the velocity increment tangentially, we will deal only with the magnitudes of the velocity, either increasing or decreasing the existing velocity, but not changing its direction. Consequently for this part we have:

Tangential Thrusting Only

$$\boxed{V_{new} = V_{old} \pm \Delta V} \qquad (1)$$

If we want to change the size of the orbit, regardless of its shape, we must change the energy of the orbit. We can look at the basic energy equation to determine the best place in the orbit to maximize our energy change for a given ΔV. Since the radius of the orbit doesn't change during the application of the ΔV we can write the change in the energy as:

$$\frac{(V + \Delta V)^2}{2} - \frac{\mu}{r} = En_{new} \qquad - \qquad \frac{V^2}{2} - \frac{\mu}{r} = En_{old} \qquad (2)$$

If we take the difference of the two we obtain the change in energy:

$$\boxed{\Delta En = V\Delta V + \frac{\Delta V^2}{2}} \qquad (3)$$

From Eq. (3) we can see that the best place to add (or subtract) energy for a given ΔV, is when V is the largest. The largest velocity occurs at the periapsis. (Recall $h = r_p V_p = r_a V_a$ = const. Hence V is greatest when r is the smallest). Consequently the most efficient place in the oribit to change energy is at the periapsis.

An Aside

Note that if we were dealing with small ΔV, we could approximate it with a differential dV. We could get the same result, to first order, by just differentiating the energy equation, holding the radius constant. For example:

$$d\left(\frac{V^2}{2} - \frac{\mu}{r} = En\right) \qquad \Rightarrow \qquad V\,dV = d\,En \tag{4}$$

Compare Eq. (4) with Eq. (3) to see that they are the same to first order in ΔV.

We can establish how the increment in velocity will affect the size of the orbit in a similar manner by writing an expression for the new and old energy in terms of the new and old semi-major axis:

$$En_{new} = -\frac{\mu}{2(a + \Delta a)} = -\frac{\mu}{2a\left(1 + \dfrac{\Delta a}{a}\right)} \qquad\qquad En_{old} = -\frac{\mu}{2a} \tag{5}$$

We can difference the two expressions to get an exact relation relating velocity increment with change in orbit size:

$$\Delta En = -\frac{\mu}{2a\left(1 + \dfrac{\Delta a}{a}\right)} + \frac{\mu}{2a} = \frac{\mu\left(\dfrac{\Delta a}{a}\right)}{2a\left(1 + \dfrac{\Delta a}{a}\right)} \tag{6}$$

or inverting we get:

$$\frac{\Delta a}{a} = \frac{2a\,\Delta En}{\mu - 2a\,\Delta En} = \frac{-En\,\Delta En}{1 + En\,\Delta En} \tag{7}$$

where $\Delta En = V\Delta V + \dfrac{\Delta V^2}{2}$.We can get a better insight if we assume we can replace the ΔV with *dV*, and the Δa with *da*. The result is obtained by just taking the differential of the Energy equation:

$$dEn = \frac{\mu}{2a^2}\,da = V\,dV \tag{8}$$

Example

Consider a circular orbit of radius r = 1 DU. We will add 20% of the circular velocity tangentially to the orbit. What are the properties of the new orbit a, e, r_a, r_p.

First lets find the properties of the current orbit: $V_c = \sqrt{\frac{\mu}{r_c}} = \sqrt{\frac{1}{1}} = 1 \text{ DU/TU}$. Then the energy and angular momentum are:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{1^2}{2} - \frac{1}{2} = -\frac{1}{2} \text{ DU}^2/\text{TU}^2 \qquad h = r V_\theta = r V = 1\,(1) = 1 \text{ DU}^2/\text{TU}$$

if we add 20% tangentially to the velocity, the new velocity will be V = 1.2 DU/TU. We can now calculate the new energy and angular momentum:

$$En = 1.\frac{2^2}{2} - \frac{1}{1} = -0.2800 \text{DU}^2/\text{TU}^2 \qquad h = 1\,(1.2) = 1.2 \text{ DU}^2/\text{TU}$$

From the energy expression we can calculate the semi-major axis:

$$En = -\frac{\mu}{2a} = -0.2800 = -\frac{1}{2a} \quad \Rightarrow \quad a = 1.7557 \text{ DU} \quad \Rightarrow \quad \Delta a = 0.7857 \text{ DU}$$

The eccentricity is obtained from:

$$e = \sqrt{1 + \frac{2 h^2 En}{\mu^2}} = \sqrt{1 + \frac{2\,(1.2)^2\,(-0.2800)}{1^2}} = 0.4400$$

The apoapsis and periapsis distance can be determined from:

$$r_a = a\,(1 + e) = 1.7847\,(1 + 0.4400) = 2.5714 \text{ DU}$$

$$r_p = a\,(1 - e) = 1.7847\,(1 - 0.4400) = 1.0000 \text{ DU}$$

As expected, since the increment in ΔV was added tangentially in the transverse direction that point had to be either an apoapsis or periapsis. Since we are making the orbit bigger, it occurred at the periapsis as was demonstrated.

Example

We will consider an elliptic orbit and add (or take away) a ΔV tangentially at the periapsis. Consider an elliptic orbit with the properties: a = 1 DU, e = 0.1. The properties of this orbit are:

$$En = -\frac{\mu}{2a} = -\frac{1}{2\,(1)} = -\frac{1}{2}\,\frac{\text{DU}^2}{\text{TU}^2} \qquad \text{Energy}$$

$$\frac{h^2}{\mu} - a(1-e^2) \quad h=\sqrt{(1)(1)(1-0.1^2)} = 0.99 \ \mathrm{DU}^2/\mathrm{TU}$$ Angular momentum

The apo and periapsis radii are:

$$r_p = a(1-e) = (1)(1-0.1) = 0.9 \ \mathrm{DU};$$
$$r_a = a(1+e) = (1)(1+0.1) = 1.1 \ \mathrm{DU}$$

We now would like to add a $\Delta V = +\ 0.1\ \mathrm{DU/TU}$ $(-0.1\ \mathrm{DU/TU})$ at the periapsis.

(Quantities in parentheses will represent the results of reducing the velocity, while those that aren't in parentheses represent adding to the velocity, only the calculations for addition are shown)

We need to calculate the new velocity a periapsis and then use that to calculate the new energy and angular momentum. First we need to calculate the velocity at periapsis of the original orbit.

$$\frac{V_p^2}{2} - \frac{\mu}{r_p} - En = -\frac{1}{2} = \frac{V_p^2}{2} - \frac{1}{0.9} \quad \Rightarrow \quad V_p = 1.1055 \ \mathrm{DU/TU}$$

The new velocity is $V_{p_{new}} = V_{p_{old}} + 0.1 = 1.2055\mathrm{DU/TU}$ $(1.0055\ \mathrm{DU/TU})$

The new energy is:

$$\frac{V_p^2}{2} - \frac{\mu}{r_p} = \frac{1.2055^2}{2} - \frac{1}{0.9} = -0.3845 \ \mathrm{DU}^2/\mathrm{TU}^2 \qquad (-0.6056\ \mathrm{DU}^2/\mathrm{TU}^2)$$

The new semi-major axis is:

$$En = -\frac{\mu}{2a} = -0.3845 = -\frac{1}{2a} \quad \Rightarrow \quad a = 1.3004 \ \mathrm{DU} \qquad (a = 0.8256 \ \mathrm{DU})$$

The new angular momentum is given by:

$$h = r_p V_p = 0.9(1.2055) = 1.08450 \ \mathrm{DU}^2/\mathrm{TU} \qquad (0.8256\ \mathrm{DU}^2/\mathrm{TU})$$

The eccentricity is obtained from:

$$e = \sqrt{1 + \frac{2h^2 En}{\mu^2}} = \sqrt{1 + \frac{2(1.0850^2)(-0.3845)}{1^2}} = 0.3079 \qquad (e = 0.0900)$$

Finally, we can compute the new apo and periapsis radii:

$$r_a = a(1 + e) = (1.3004)(1 + 0.3079) = 1.7008 \text{ DU} \qquad (0.9000\text{DU})$$

$$r_p = a(1 - e) = (1.3004)(1 - 0.3079) = 0.9000 \text{ DU} \qquad (0.7513\text{DU})$$

One thing that we can learn from the previous problem is that if we add velocity at the periapsis, we will raise the apoapsis, and if we remove velocity at the periapsis, we will lower the apoapsis. Hence a periapsis burn will raise (lower) the apoapsis, but the periapsis radius will stay the same. It is easily shown that if we add or (remove) velocity at the apoapsis, we will increase or raise (decrease or lower) the periapsis radius.

Delta-V Required to Change Orbit to a Specified Size

The problem we would like to address here is that of determining the amount of ΔV required to raise or lower apoapsis (or periapsis) a specified amount. Here we will specify an initial circular orbit such that the periapsis radius of the new orbit is the same as the circular radius of the original orbit, $r_p = r_c$. The apoapsis height of the new orbit will be specified as r_a.
In order to determine the required ΔV, we need to determine the velocities at the periapsis and apoapsis of a generic elliptic orbit. We have already done this calculation, but will repeat it here. To calculate the velocity at any point in an orbit, we use the energy equation:

$$En = -\frac{\mu}{2a} = -\frac{\mu}{r_p + r_a} = \frac{V^2}{2} - \frac{\mu}{r} \tag{9}$$

We can evaluate the energy equation at the periapsis and at the apoapsis in order to get the velocities at these two points:

$$\boxed{V_P = \sqrt{\frac{\mu}{r_p}}\sqrt{\frac{2\dfrac{r_a}{r_p}}{1 + \dfrac{r_a}{r_p}}} \qquad V_a = \sqrt{\frac{\mu}{r_a}}\sqrt{\frac{2}{1 + \dfrac{r_a}{r_p}}}} \tag{10}$$

Now, if we are in a circular orbit, at radius $r_c = r_p$, we already have a transverse velocity of $V_c = \sqrt{\frac{\mu}{r_c}} = \sqrt{\frac{\mu}{r_p}}$. Then the increment in velocity is given by V_p in the new orbit minus V_c in the original circular orbit. The increment in velocity is the velocity that I want minus the velocity that I have. Generally this is a vector difference (that we will discuss later), but here, since the two velocities are in the same direction, is just an algebraic difference. Consequently we have

Delta-V required to raise apoapsis from a circular orbit

$$\Delta V = V_{required} - V_{circular} = V_p - V_c = \sqrt{\frac{\mu}{r_p}}\left[\sqrt{\frac{2\frac{r_a}{r_p}}{1 + \frac{r_a}{r_p}}} - 1\right] \qquad (11)$$

We can calculate the required ΔV to lower the periapsis form a circular orbit. In this case the radius of the circular orbit is the apoapsis radius of the new orbit. Again, we take the velocity that we want, the velocity at apoapsis of the new orbit, and subtract from it the velocity of the original circular orbit. In this case $r_a = r_c$, and r_p is the new specified periapsis radius.

Delta V require to lower periapsis from a circular orbit

$$\Delta V = \sqrt{\frac{\mu}{r_a}}\left[\sqrt{\frac{2}{1 + \frac{r_a}{r_p}}} - 1\right] \qquad (12)$$

If we do this last calculation for a specific problem, we will see that ΔV is negative. All this means is that the increment of velocity is in the opposite direction of the original velocity, and that the vehicle is slowing down. Generally when we discuss the velocity increment, ΔV, we are interested in the magnitude of it, since its value is related to the amount of fuel burned.

Orbit Transfer

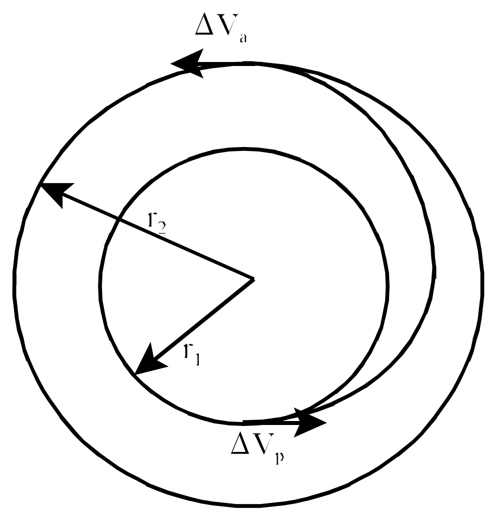
The previous section indicated how we could change from one orbit to another by incrementing the velocity tangentially. Consequently, the two orbits, the original and the new one are tangent to each other at some point. However, if we perform such a maneuver twice, we can change from one orbit to another orbit that doesn't intersect the first. In particular we can use this method to transfer between tow circular orbits. The intermediate orbit that joins the two circular orbits is called the transfer orbit, and if it is tangent to both circular orbits is called a Hohmann transfer orbit.

The Hohmann Transfer

The Hohmann transfer orbit between two circular orbits of radius r_1 (inner) and r_2 (outer) is an orbit that is tangent to both circular orbits. Hence we have:

$$r_p = r_1 \qquad\qquad r_a = r_2 \qquad (13)$$

The Hohmann transfer orbit between two circular orbits is shown in the sketch to the right. Here we are in the inner orbit of radius r_1 and add ΔV_p to the original circular speed to enter the periapsis of the transfer orbit (or raise the apoapsis). If we did nothing more, we would remain in that elliptical transfer orbit, However, when we arrive at the apoapsis of the transfer orbit, we add another ΔV_a to increase our velocity to the circular speed of the outer orbit (or raise the periapsis). Such increment of velocity will put us in the outer circular orbit - transfer complete. The increments of velocity at the two tangent points can be computed using the results that we developed previously. We can then sum the magnitudes of the two ΔVs to get the total ΔV required for the transfer. These calculations follow (recall Eq, (23))



$$\Delta V_p = V_{p_T} - V_{c_1} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2\frac{r_2}{r_1}}{1 + \frac{r_2}{r_2}}} - 1\right)$$

$$\Delta V_a = V_{c_2} - V_{a_T} = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2}{1 + \frac{r_2}{r_1}}}\right) \tag{14}$$

The total ΔV for the transfer is given by:

$$\Delta V_{total} = |\Delta V_p| + |\Delta V_a| \tag{15}$$

A transfer from the outer orbit to the inner orbit would require the exact same ΔV. However the velocity increments would be in the opposite direction as we would be taking energy out of the system to reduce its size.

Example

Consider a fight from Earth's orbit about the Sun to the planet Uranus' orbit about the

Sun. Note, we are going from one orbit to the other (not from planet to planet). We will assume that both obits are circular with the radii given in Sun distance units called astronomical units (AU). The Earth is in an orbit of 1 AU, and Uranus is in an orbit of 19.28 AU. What ΔV is required to transfer from one orbit to the other? The transfer orbit will be an elliptic orbit about the sun that is tangent to both the Earth and Uranus orbits. Hence its aphelion radius is $r_a = 19.28 \text{ AU}$, and its perihelion radius is $r_p = 1 \text{ AU}$.

At perihelion:

$$V_{p_T} = \sqrt{\frac{\mu}{r_1}} \sqrt{\frac{2\frac{r_2}{r_1}}{1 + \frac{r_2}{r_1}}} = \sqrt{\frac{1}{1}} \sqrt{\frac{2\,(19.28}{1 + 19.28}} = 1.3789 \text{ AU/TU}$$

$$V_{c_1} = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{1}{1}} = 1 \text{ AU/TU}$$

$$\Delta V_p = V_{p_T} - V_{c_1} = 1.3789 - 1.0 = 0.3789 \text{ AU/TU}$$

At aphelion:

$$V_{a_T} = \sqrt{\frac{\mu}{r_2}} \sqrt{\frac{2}{1 + \frac{r_a}{r_2}}} = \sqrt{\frac{1}{19.28}} \sqrt{\frac{2}{1 + 19.28}} = 0.0175 \text{ AU/TU}$$

$$V_{C_2} = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{\mu}{19.28}} = 0.2277 \text{ AU/TU}$$

$$\Delta V_a = V_{c_2} - V_{a_T} = 0.2277 - 0.0715 = 0.1562 \text{ AU/TU}$$

Total Delta-V

$$\Delta V_{tot} = |\Delta V_p| + |\Delta V_a| = 0.3789 + 0.1562 = 0.5351 \text{ AU/TU}$$

$$= 0.5351 \cdot 29.7848 \text{ km/s/AU/TU} = 15.9378 \text{ km/s}$$

Compare this value with the ΔV required to escape from the solar system.

For escape, we need to enter into a parabolic orbit when we leave the Earth's orbit

$$\Delta V_{escape} = V_{escape} - V_{c_1} = \sqrt{\frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}} = (\sqrt{2} - 1)\sqrt{\frac{1}{1}} = 0.4142 \text{ AU/TU}$$

Hence the total fuel requirement to transfer from Earth's orbit to Uranus' orbit is greater than that required to leave the solar system!

Transfer time for Hohmann Orbit

Even though we haven't introduce the time variable into the problem, we can still calculate the time for a Hohmann transfer. Since the orbit properties are symmetric with the semi-major axis, we can see that it will take the same amount of time to travel from periapsis to apoapsis as it does from apoapsis to periapsis. Therefore the time it takes for the Hohmann transfer is half the period of the orbit:

Time for Hohmann Transfer

$$TOF = \frac{T_p}{2} = \pi\sqrt{\frac{a^3}{\mu}} \tag{16}$$

Example

For our trip to Uranus in the previous example we can compute the travel time from Eq. (16).

$$TOF = \pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\left(\frac{1 + 19.28}{2}\right)^3 \frac{1}{1}} = 101.4394 \; TU_{sun}$$

$$101.4394 \cdot 58.1328 \text{ days/TU}_{\text{sun}} = 5896.956 \text{ days} = 16.15 \text{ years}$$

Time and Phase Angle to Launch

In the previous section we determined the travel time and Delta-V associated with transferring between two circular orbits using a Hohmann transfer orbit. If we want to travel from planet to planet (or intercept or rendezvous with a satellite), we must be sure that we time our launch just right so that the planet (or satellite) will be there when we arrive. Consequently we need to determine the time to wait until launch or alternatively, the relative position of the two planets or satellites (phase angle) at launch. There are several ways to calculate these values. The method presented here requires the least amount of intuition and will always give the correct result regardless if the problem requires going from inner to outer orbits or vice-versa. Consider the initial planet or satellite to be (1)1, and the target planet or satellite to be (2). We will make use of the facts that we know the time of transfer, and that the arrival point is 180 degrees away from the launch point. With these two bits of information we can determine the time to wait or the phase angle at launch.

Time to Launch

We can pick some reference direction from which to measure all angles. It will be assumed that we know the initial positions of both satellite (1) and (2) at the current time that we will designate as the epoch, t_0. The departure from satellite (1) will be designated as the launch time, t_{L_1}. Then the position at launch of satellite (1) is:

$$\theta_1(t_L) = \theta_1(t_0) + n_1 (t_{L_1} - t_0) \tag{17}$$

where n_1 is the angular rate of satellite (1). The position of satellite (2) at the time of arrival $(t_L + TOF)$ is then given by:

$$\theta_2(t_L + TOF) = \theta_2(t_0) + n_2 (t_L - t_0) + n_2 (TOF) \tag{18}$$

where n_2 is the angular rate of satellite (2). The difference of the two positions must be some odd multiple of 180 degrees or π. Hence we can write:

$$\theta_2(t_L + TOF) = \theta(t_{L_1}) + (1 + 2k)\pi \tag{19}$$

where k is some positive or negative integer. From Eq. (17) and (18) we can write Eq. (19) as:

$$\theta_2(t_0) + n_2 (t_L - t_0) + n_2 (TOF) = \theta_1(t_0) + n_1 (t_L - t_0) + (1 + 2k)\pi \tag{20}$$

Equation (20) can be rearranged to give us some insight into the problem:

$$[\theta_2(t_0) - \theta_1(t_0)] + n_2 (TOF) = (n_1 - n_2)(t_L - t_0) + (1 + 2k)\pi \tag{21}$$

1 2 3 4

We can examine the terms in Eq. (21). The first term is the relative positions of the two satellites at epoch (phase angle at epoch). The second term is just the angle the target satellite goes through during the transfer. The transfer time-of-flight (TOF) is known. The term marked 3 is the relative angular rate between the two satellites. The term marked 4 is the time-to-wait, the launch time minus the epoch time. Hence we can solve for the time-to-wait:

Time to Wait for Launch

$$(t_{L_1} - t_0) = \frac{[\theta_2(t_0) - \theta_1(t_0)] + n_2(TOF) - (1 + 2k)\pi}{n_1 - n_2} \qquad (22)$$

where k is a positive or negative integer selected to make the time to launch the smallest positive number. That time-to-wait would be the first opportunity to launch. The next value of k would be the next possible launch time, etc, each subsequent integer being the next possible launch time.

An alternative way of thinking about the launch time is to determine the angles between the two satellites when it is appropriate to launch, of the phase angle at launch. We can determine this angle from Eq. (22). We simply define the epoch time to be the launch time, $t_0 = t_{L_1}$ and solve for the phase angle at launch, $\theta_2(t_0) - \theta_1(t_0)$:

Phase Angle at Launch

$$(\theta_2 - \theta_1)_{launch} = -n_2(TOF) + (1 + 2k)\pi \qquad (23)$$

where again, k may be selected so that the phase angle has a magnitude of less than 2π. It can be positive or negative at your discretion. Note that the phase angle is defined as the angle to the target satellite minus the to the launch satellite so that a positive phase angle means that the target satellite leads the launch satellite by the phase angle amount.

Example

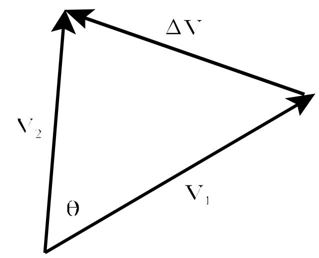
What is the phase angle required for the trip to Uranus?

$$\begin{aligned}
(\theta_2 - \theta_1)_{launch} &= -n_2(TOF) + (1 + 2k)\pi \\
&= -\sqrt{\frac{1}{r_2^3}}(TOF) + (1 + 2k)\pi \\
&= -\sqrt{\frac{1}{19.28^3}}(101.4394) + (1 + 2k)\pi \\
&= -1.1982 + (1 + 2k)\pi \qquad k=0 \\
&= 1.9434 \text{ rad} \\
&= 111.348 \text{ deg}
\end{aligned}$$

At launch from Earth's orbit, Uranus leads Earth by 111.348 deg.

Applying Delta-V Non-Tangentially

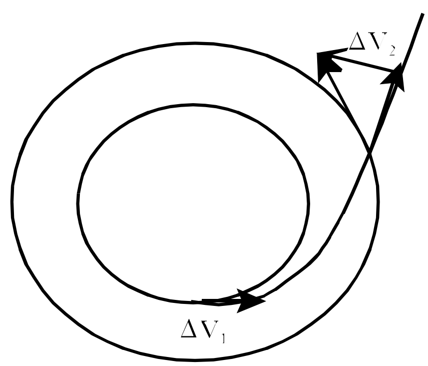
Up to this point we have only consider applying the velocity increment tangentially. For all practical purposes, that is the most efficient way to gain or lose energy. However, there are other considerations that sometime require us change to an orbit that is not tangent to the original one. We can compute the required ΔV in precisely the same manner that we have previously, except we now need to remember that we are considering vectors. Suppose that we have a velocity in our original orbit designated by $\vec{V}_1$, and at the same point (remember the velocity is changed instantaneously) we desire to be at a velocity $\vec{V}_2$ in the new orbit. The change in velocity is just the vector difference of the two: $\Delta\vec{V} = \vec{V}_2 - \vec{V}_1$, and is a vector. For now, however, we are interested in the magnitude of the $\Delta\vec{V}$ for fuel requirement calculations. A typical scenario is shown in the figure where we have the velocity in the original orbit, $\vec{V}_1$, the desired velocity in the new orbit, $\vec{V}_2$, and the increment in velocity required to change velocities, $\Delta\vec{V}$. If the angle between the two velocities is θ, then the magnitude of $\Delta\vec{V}$, is given by the law of cosines:



$$\boxed{(\Delta V)^2 = V_1^2 + V_2^2 - 2\,V_1\,V_2\cos\theta \qquad (24)}$$

where $\Delta V = |\Delta\vec{V}|$, $V_1 = |\vec{V}_1|$, and $V_2 = |\vec{V}_2|$. Note that the direction of the difference of two vectors can be remembered by the simple statement "head - tail." The difference vector is vector at the head end minus the vector at the tail end.

Example



Consider a parabolic transfer orbit from Earth's orbit to the orbit of Uranus. The transfer orbit will be tangent to the Earth's orbit a perihelion, and it will intersect the orbit of Uranus.
The inner orbit is the Earth orbit with $r_1 = 1$ AU. The outer orbit is Uranus' orbit, $r_2 = 19.28$ AU. The initial ΔV_1 is determined by noting the burn is tangential to the original circular orbit and changes the velocity from circular velocity to escape velocity (the velocity in a parabolic orbit).

$$\Delta v_1 = V_{esc} - V_c = \sqrt{\frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}}$$

$$\Delta V_1 = (\sqrt{2} - 1)\sqrt{\frac{1}{2}} = 0.4142 \text{ AU/TU}$$

Arrival at Uranus' orbit:

The transfer orbit is a parabola, so the equation for the orbit is $r = \dfrac{h^2/\mu}{1 + \cos\nu}$.

Further, we can recall that the periapsis radius is $r_p = h^2/(2\mu) = 1$ AU. Then we can write the equation of the transfer orbit as

$$r = \frac{2}{1 + \cos\nu} = 19.28 \qquad \Rightarrow \qquad \cos\nu = -0.8962 \qquad \Rightarrow \qquad \nu = \pm 153.671 \text{ deg}$$

The flight path angle at arrival is given by:

$$\tan\phi = \frac{\sin\nu}{1 + \cos\nu} = \tan\nu/2 \qquad \Rightarrow \qquad \phi = \frac{\nu}{2} = 153.671/2 = 76.838 \text{ deg}$$

The velocity at arrival is just $V_{trans} = \sqrt{\dfrac{2\mu}{r}} = \sqrt{\dfrac{2(1)}{19.28}} = 0.3221$ AU/TU

and the circular velocity of Uranus is:

$$V_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{1}{19.28}} = 0.2277 \text{ AU/TU}$$

The angle between the desired circular velocity direction and the arrival parabolic orbit velocity direction is just the flight path angle (the circular orbit velocity is at zero flight path angle). Hence we can now use the law of cosines to determine the magnitude of the $\Delta\vec{V}_2$.

$$(\Delta V_2)^2 = V_{trans}^2 + V_{cir}^2 - 2\, V_{trans} V_{cir}^2 \cos\phi$$

$$= (0.3221)^2 = (0.2277)^2 - 2\,(0.3221)\,(0.2277)\cos(76.838)$$

$$= 0.3496 \text{ AU/TU} = 10.413 \text{ km/s}$$

The total ΔV for the trip is just the scalar sum of the two:

$$\Delta V_{tot} = 0.4142 + 0.3496 = 0.7638 \text{ AU/TU} = 22.74 \text{ km/s}$$

Note that this value is significantly larger than for the Hohmann transfer calculated previously

(0.5351 AU/TU).

We can extend the ideas presented previously to calculate a complete round trip say from Earth to Mars and return. If we assume some starting location for the planets, we can compute the wait time, the transfer time, the wait time for alignment for return, and the return travel time. The result will be a log of the trip. We will consider a round trip to Mars in the following example:

Example - Round trip from Earth to Mars and Return using Hohmann transfers.

In this example Mars will be considered the "target' planet and Earth the "launch' planet. We will assume that each planet is in a circular orbit, and that we will use a Hohmann transfer orbit to go from Earth to Mars and return. The properties of the two planet orbits are:

$$r_1 = r_{Earth} = r_p = 1 \text{ AU}$$

$$V_{c_1} = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{1}{1}} = 1 \text{ AU/TU}$$

$$r_2 = r_{Mars} = r_a = 1.524 \text{ AU}$$

$$V_{c_2} = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{1}{1.524}} = 0.8100 \text{ AU/TU}$$

At perihelion we have:

$$\Delta V_p = V_{p_{tran}} - V_{c_1} = \sqrt{\frac{\mu}{r_p}}\left[\sqrt{\frac{2\frac{r_a}{r_p}}{1 + \frac{r_a}{r_p}}} - 1\right]$$

$$= \sqrt{\frac{1}{1}}\left[\sqrt{\frac{2\,(1.524)}{1 + 1.524}} - 1\right] = 1.0989 - 1$$

$$= 0.0989 \text{ AU/TU} \cdot 29.7848 \text{ km/s/AU/TU} = 2.9461 \text{ km/s}$$

At aphelion we have:

$$\Delta V_a = V_{c_2} - V_{a_{trans}} = \sqrt{\frac{\mu}{r_a}}\left[1 - \sqrt{\frac{2}{1 + \frac{r_a}{r_p}}}\right]$$

$$= \sqrt{\frac{1}{1.524}}\left[1 - \sqrt{\frac{2}{1 + 1.524}}\right]$$

$$= 0.0890 \text{ AU/TU} \cdot 29.7848 \text{ km/s/AU/TU} = 2.6500 \text{ km/s}$$

$$\text{Total } \Delta V = \Delta V_p + \Delta V_a = 0.0989 + 0.0890 = 0.1879 \text{ AU/TU} = 5.5960 \text{ km/s}$$

Time of flight (TOF)

$$TOF = \pi\sqrt{\frac{a_{trans}^3}{\mu}} = \pi\sqrt{\frac{(r_p + r_a)^3}{8\,\mu}} = \pi\sqrt{\frac{(1 + 1.524)^3}{8\,(1)}} = 4.4539 \text{ TU}$$

$$= 4.4539 \cdot 58.1328 \text{ days/TU} = 258.92 \text{ days} = 0.7089 \text{ years}$$

Wait time to launch:

For the purpose of this example, we can assume that at epoch, Mars and Earth are aligned on the same side of the Sun (conjunction). Hence the phase angle at epoch will be zero. The equation for the time to wait is given by:

$$(t_{L_1} - t_0) = \frac{[\,\theta_2(t_0) - \theta_1(t_0)\,] + n_2\,(TOF) - (1 + 2k)\,\pi}{n_1 - n_2}$$

we need additional information to evaluate this equation, in particular n_1 and n_2..

$$n_1 = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{1}{1^3}} = 1 \text{ rad/TU} \qquad n_2 = \sqrt{\frac{\mu}{a_2^3}} = \sqrt{\frac{1}{1.524^3}} = 0.5315 \text{ rad/TU}$$

The time to wait becomes:

$$(t_{L_1} - t_0) = \frac{[0] + 0.5315\,(4.4539) - (1 + 2k)\pi}{1 - 0.5315} = 11.7586 \text{ TU} \qquad (k = 1)$$

$11.7586 \text{ TU} = 683.56 \text{ days} - 1.8715 \text{ years}$ Hence from that particular epoch, we would have to wait nearly two years before the planets aligned for a minimum fuel Hohmann transfer from Earth to Mars!

From here on out, we will reckon trip time from the launch time. So the new epoch will be t_{L_1} . We can set $t_0 = t_{L_1}$ and substitute into the wait-time equation and solve for the phase angle at launch to get

$$\begin{aligned}(\theta_2 - \theta_1)_{launch} &= -n_2\,(TOF) + (1 + 2k)\pi \\ &= -0.5315\,(4.4539) + (1 + 2k)\pi \\ &= -2.3673 + \pi \qquad (k{=}0) \\ &= 0.7742 \text{ rad} = 44.3612 \text{ deg}\end{aligned}$$

Hence Mars must lead the Earth by 44.36 degrees when the vehicle is launched from Earth in order for Mars to be in the right place when it arrives at Mars' orbit.

From here on out, we will measure angles from the Earth's position at epoch (launch time). Under this convention, $\theta_1(t_{L_1}) = \theta_1(t_0) = 0$, and $\theta_2(t_{L_1}) = \theta_2(t_0) = 0.7742 \text{ rad}$.

Arrival at Mars (t = t$_2$)

We can calculate the position of the planets at the arrival time at Mars. The basic equation is $\theta(t) = \theta(t_{L_1}) + n\,(t - t_{L_1})$. The planet positions are given by

$$\theta_1(t_2) = \theta_1(t_{L_1}) + n_1\,(TOF) = 0.0 + 1\,(4.4539) = 4.4539 \text{ rad}$$

$$\theta_2(t_2) = \theta_2(t_{L_1}) + n_2\,(TOF) = 0.7742 + 0.5315\,(4.4539) = 3.1416 \text{ rad}$$

(Note we knew that Mars was at π even without doing the last calculation!)

The phase angle at Mars' arrival is just the difference of the two:

$$\begin{aligned}(\theta_2 - \theta_1)_{t_2} &= 3.1416 - 4.4539 = -1.3123 \text{ rad} + 2\pi = 4.9709 \text{ rad} \\ &= -75.1888 \text{ deg} + 360 = 284.8112 \text{ deg}\end{aligned}$$

Hence at arrival at Mars, Mars is 75.2 degrees behind the Earth, or Mars leads the Earth by 284.8

deg, whichever way you desire to think about it.

Wait time on Mars

When it is time to leave Mars, we want the Earth to be located so that after the transfer back to Earth's orbit, the Earth will be in the right place to receive us. Since we are using a Hohmann transfer to return, then we know that when we arrive back on Earth, it must be 180 degrees from where we launched from Mars. If we reckon time from Earth launch, we can calculate the wait time on Mars in the following way. First lets designate the wait time on Mars as the time to launch from Mars, t_{L_2}, minus the time to arrive at Mars, t_2, $t_{wait} = t_{L_2} - t_2$. We can define the arrival time at Earth as t_4. Now we can calculate the position of Mars at launch form Mars, and the position of the Earth at arrival:

$$\theta_2(t_{L_2}) = \theta_2(t_{L_1}) + n_2\,(TOF + t_{wait})$$
$$\theta(t_4) = \theta_1(t_{L_1}) + n_1\,(2\,TOF + t_{wait})$$

The difference of these two should be some odd multiple of π.

$$(1 + 2k_2)\,\pi = \theta_1(t_4) - \theta_2(t_{L_2}) = (\theta_1 - \theta_2)_{t_{L_1}} + (2n_1 - n_2)\,TOF + (n_1 - n_2)\,t_{wait}$$

We can solve this equation for the time-to-wait

Time to Wait

$$t_{wait} = \frac{(1 + 2k_2)\,\pi - \left[(\theta_1 - \theta_2)_{t_{L_1}} + (2n_1 - n_2)\,TOF\right]}{n_1 - n_2} \qquad (24)$$

where we select the integer k_2 to give us the smallest positive wait time.

Substituting in the numbers we have: ($k_2 = 1$)

$$t_{wait} = \frac{(1 + 2k_2) - \left[-0.7742 + (2(1) - 0.5315)(4.4539)\right]}{1 - 0.5315} = 7.8096 \text{ TU} = 453.99 \text{ days}$$

Note that if we picked $k_2 = 2$, that would give us the wait time for the next opportunity to launch.

Positions and Phase Angle at Launch from Mars (t_{L_2})

We can easily compute the phase angle at Mars launch by calculating the position of the planets at that time:

$$\theta_2(t_{L_2}) = \theta_2(t_{L_1}) + n_2\,(TOF + t_{wait}) = 0.7742 + 0.5815\,(4.4539 + 7.8096)$$
$$= 7.2926 \text{ rad} = 417.8333 \text{ deg}$$
$$\theta_1(t_{L_2}) = \theta_1(t_{L_1}) + n_1\,(TOF + t_{wait}) = 0 + (1)\,(4.4539 + 7.8096)$$
$$= 12.2635 \text{ rad} = 702.6446 \text{ deg}$$

The phase angle is just the difference:

$$(\theta_2 - \theta_1)_{t_{L_2}} = 7.2926 - 12.2635 = -4.9709 \text{ rad} + 2\pi = 1.3123 \text{ rad} = 75.1888 \text{ deg}$$

Note that here we still retain the phase angle as the initial “target” planet minus the initial “launch” planet, even though their roles have been reversed. So at this point, Mars is leading the Earth by 75.2 degrees.

Positions and Phase Angle at Arrival at Earth (t_4)

In the same manner we can calculate the positions and phase angle at Earth arrival.

$$\theta_2(t_4) = \theta_2(t_{L_1}) + n_2\,(2\,TOF + t_{wait}) = 0.7742 + 0.5815\,((2)\,(4.4539) + 7.8096)$$
$$= 9.6599 \text{ rad} = 553.4722 \text{ deg} = 193.4722\ ^\circ$$
$$\theta_1(t_4) = \theta_1(t_{L_1}) + n_1\,(2\,TOF + t_{wait}) = 0 + (1)\,((2)\,(4.4539) + 7.8096)$$
$$= 16.7173 \text{ rad} = 957.8333 \text{ deg} = 237.8333\ ^\circ$$

The phase angle is determined by taking the difference

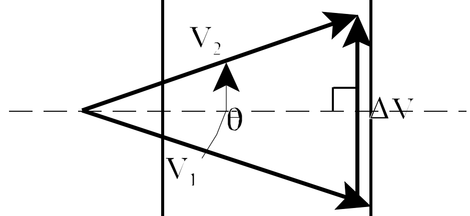
$$(\theta_2 - \theta_1)_{t_4} = 9.6599 - 16.7173 = -7.0574 \text{ rad} + 2\pi = -0.7742 \text{ rad} = -44.3612 \text{ deg}$$

Hence Earth leads Mars by 44.36 degrees, or Mars is 44.36 degrees behind the Earth. The astute reader will note some symmetry (or asymmetry) in the problem. At launch Mars lead Earth by 44.36 degrees. A table indicating positions times and phase angles follows.

Trip Log

	Time(TU)	Days	Earth position (deg)	Mars position (deg)	Phase Angle (deg)
t_{L_1}	0	0	0	44.46	44.36

+TOF	4.4539				
t_2	4.4539	258.92	255.19	180.00	-75.19
$+t_{wait}$	7.8096				
t_{L_2}	12.2635	712.91	702.64 (342.64)	417.83 (57.83)	75.19
+TOF	4.4539				
t_4	16.7173	971.83 (2.66 yrs)	957.83 (237.83)	553.47 (193.47)	-44.36



$\Delta V_p = 2.9461 \text{ km/s} = 0.0989 \text{ AU/TU}$ $\Delta V_a = 2.6500 \text{ km/s} = 0.0890 \text{ AU/TU}$

$\Delta V_{total} = 5.5961 \text{ km/s} = 0.1879 \text{ AU/TU}$

Orbit Plane Change Maneuver

All the maneuvers discussed previously have been in the same original orbit plane. If we would like to keep the same identical orbit, but change the plane in which it is located, we can use a pure plane change maneuver. Again, all we need to do is remember we are dealing with vectors and find the difference of the two vectors. In this case the two velocities are equal but in different directions. In the figure to the right, we are looking edgewise at the orbit, right down the position vector. The original orbit has the velocity $\vec{V}_1$ that lies in the plane of the orbit. We would like to change the plane through an angle θ. The velocity in the new orbit plane is equal in magnitude, but now lies in the new orbit plane. The question is, what is the velocity increment, $\Delta\vec{V}$ required to make this plane change. Again, for now we are only interested in the magnitude, $\Delta V = |\Delta\vec{V}|$. Also we have $|\vec{V}_1| = |\vec{V}_2| = V$. Then, from the geometry we have

Delta-V for pure plane change (no change in V)

$$\frac{\Delta V}{2} = V \sin\frac{\theta}{2} \quad \Rightarrow \quad \boxed{\Delta V = 2\, V \sin\frac{\theta}{2}} \qquad (25)$$

We could also do a plane change that took us from one orbit to a different orbit in a different plane. The restriction here as that we do it with a single burn. Under these circumstances, we can draw a similar figure, but the initial and final velocities will be different, and at some angle θ apart. The figure will be similar to the non-tangential burn we studied earlier. Consequently, we can just write down law of cosines:

Delta-V for plane change with two different velocity magnitudes

$$\Delta V^2 = V_1^2 + V_2^2 - 2\, V_1 V_2 \cos\theta \qquad (26)$$

Consider the following problem; We would like to change orbits from a small circular orbit (a low earth orbit) to a large circular orbit in a different plane (geosynchronous orbit). For example we launch from Cape Canaveral into the low Earth orbit and would like to change to an equatorial geosynchronous orbit. There are several strategies that we can use to perform this orbit change:

1) Perform a small circle to large circle Hohmann transfer in the original plane, and then perform a pure plane change.

2) Perform a pure small circular orbit plane change, then perform the Hohmann transfer

3) Perform a plane change to the new plane such that the velocity in the new plane is that of a Hohmann transfer.

4) Perform a Hohmann transfer, but make the apogee burn one that performs the required plane change and ends up with the required velocity for a large circular orbit.

Each of these scenarios will require a different amount of total ΔV, one providing the minimum. You should be able to determine which provides the most fuel efficient strategy by considering Eqs. (25) and (26).

Converting ΔV into Fuel Consumption

All the above maneuvers and orbit changes required a certain amount of ΔV. The results were independent of spacecraft size. We would now like to find out what the required fuel usage would be to provide that ΔV. One would expect the result to depend on vehicle mass. We use the so-called "rocket equation"

$$T = -\dot{m}\, V_e = m \frac{dV}{dt} \qquad (27)$$

where T = thrust

m	=	vehicle mass
V_e	=	exhaust exit velocity
V	=	velocity

The thrust is proportional to the rate that mass is flowing out of the vehicle and the exit velocity of the rocket engine. We can rearrange Eq. (27) in a manner that allow us to integrate it.

$$dV = -V_e m \frac{\cot}{m} dt = -v_e \frac{dm}{m}$$

$$V_2 - V_1 = -V_e \ln m \Big|_{m_1}^{m_2} = -V_e \ln \frac{m_2}{m_1} \qquad (28)$$

$$\Delta V = V_e \ln \frac{m_1}{m_2}$$

where	m_1	=	initial mass of the vehicle before the burn
	m_2	=	final mass of the vehicle after the burn
	V_e	=	engine exhaust (or exit) velocity
	ΔV	=	magnitude of velocity change

The exit velocity of a rocket is related to a quantity called the specific impulse, I_{sp} in the following way

$$\boxed{V_e = g_0 I_{sp} \qquad (29)}$$

where g_0 is the sea-level value of the acceleration due to gravity. The difference in the initial and final mass is the mass of the fuel burned. Hence we can write Eq. (28) in an alternative form that contains the mass of the fuel directly:

$$\boxed{\Delta V = g_0 I_{sp} \ln \frac{m_2 - m_{fuel}}{m_2} = g_0 I_{sp} \ln \frac{m_1}{m_1 - m_{fuel}} \qquad (30)}$$

Example

Consider a vehicle with an initial mass of 136 kg. Calculations show a ΔV= 1 DU/TU = 7.9054 km/s = 7905.4 m/s is required for some maneuver. The engine is has an

I_{sp} = 400 s.   How much fuel is required?

$$\Delta V = 7905.4 = g_0 I_{sp} \ln \frac{m_0}{m_0 - m_{fuel}} = (9.8066)(400) \ln \frac{136}{136 - m_{fuel}}$$

$$\ln \frac{136}{136 - m_{fuel}} = 2.0153, \qquad \frac{136}{136 - m_{fuel}} = 7.5030, \qquad m_{fuel} = 117.87 \text{ kg}$$

$$\frac{m_{fuel}}{m_1} = \frac{117.87}{136.0} = 0.87$$ The ratio of fuel mass over initial mass is called the ***fuel fraction***
