7. Mission Analysis (Patched Conic Approach)

By mission analysis we generally mean determining the fuel and time budget for carrying out an interplanetary mission. Although this problem can be difficult, we can simplify it by using the idea of the patched conic approximation. For this approach we consider the problem to be broken up into several two body problems, that are patched together. This idea comes for the concept of the "sphere of influence." The sphere of influence is considered to be an imaginary sphere about the smaller of two (or more) bodies inside of which only the attractive force of the small body is considered so that the orbits may be considered to be those of a "two body" problem. Once outside the sphere, the problem is considered to be a "two body" problem with the attractive force due only to the larger body. At the sphere of influence, the two solutions are "patched" together. For example if we consider an Earth - Mars mission. The transfer orbit from Earth's orbit to Mars' orbit may be considered to be the solution to the Sun - Vehicle two body problem, a heliocentric orbit. In the neighborhood of the Earth, however, we will ignore the Sun effects and consider the launch ascent ellipse, the Earth parking orbit, and the Earth escape hyperbolic orbit to be geocentric two body orbits, ignoring the effects of the Sun. We make the same assumption upon arrival at Mars, that the entry hyperbolic orbit, the capture orbit, the swing-by orbit, and any other desired orbit are Mars-centered two body orbits neglecting the effects of the Sun (and Earth). The transition from heliocentric to Mars or Earth-centered orbits occurs at the sphere of influence boundary. However for these interplanetary mission analysis calculations, the following rule applies: From the point of view of the Sun, the sphere of influence about any planet has zero radius (changes happen at the planet orbit radius), while from the point of view of the planets, the sphere of influence is infinite, that is the properties of orbits at the boundary of the sphere of influence are the properties at infinity with respect to the planet. Consequently, for these calculations, we do not need to know the radius of the sphere of influence.

However, for completeness, we can calculate the sphere of influence from the following equation:

$$r_s = \left(\frac{m_1}{m_2}\right)^{2/5} r_{12} \tag{1}$$

Here, r_{12} is the distance between the two bodies, m_1 is the mass of the smaller body (say Earth) and m_2 is the mass of the larger body (say Sun). r_s Is the radius of the sphere of influence. We can consider the Earth-Moon system where there would be a sphere of influence about the moon (with respect to the Earth) which would fully lie inside the sphere of influence of the Earth, with respect to the Sun. In problems concerning the Moon, one may not be able to ignore the radius of the sphere of influence of the moon when calculating Earth-Moon trajectories. We can, however ignore the planet spheres of influence when doing mission analysis regarding the planets.

The mission analysis problem always starts by calculating the Heliocentric orbit(s) first. For example if we were to use a Hohmann transfer orbit, we would calculate the required velocities and ΔVs as required. Here we will carry an example through as we explore each portion of the mission. The example that we will use is a mission to Mars, either an orbiter or a fly-by, with the transfer orbit tangent to Earth's orbit, but in a heliocentric orbit with a two year period. The properties of this heliocentric transfer orbit are easily found to be:

a_T = 1.5874 AU, e = 0.3700 r_p = 1 AU $V_1 = V_p$ = 1.1705 AU/TU

7.1 Heliocentric Orbit

The first calculation to be done when performing mission analysis is to calculate the heliocentric orbit properties. Generally it is necessary to determine the energy and angular momentum from the information given. Once these quantities are known, the conditions at the Earth's orbit and at the target planet's orbit can be determined. Assuming these locations are designated (1) and (2) respectively, we can find the speeds, V_1 and V_2 from the ***energy equation***:

$$\frac{V_i^2}{2} - \frac{\mu}{r_i} = En \qquad i = 1,2 \tag{2}$$

And the flight path angle from the ***angular momentum:***

$$h = r_i V_i \cos\phi_i \tag{3}$$

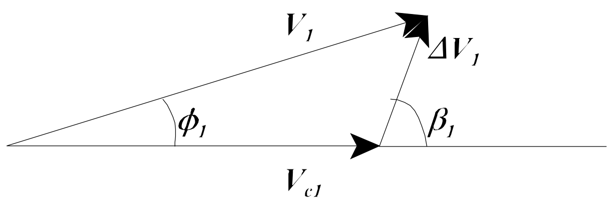
Hence from the heliocentric orbit properties we can get V_1, ϕ_1, V_2 and ϕ_2. Since we know the orbital speeds of the Earth and the target planet, we can calculate the necessary ΔV at the departure point (1) and at the arrival point (2) necessary to leave the Earth's orbit or to enter into the target planet's orbit, ΔV_1 **and** ΔV_2 respectively. In the mission analysis problem, these ΔVs convert into the velocity relative to the respective planets or V_∞ with respect to the planet. This result can be established in the following way: If a vehicle were to escape Earth (or any other planet) using a parabolic orbit, when it got to infinity, its velocity relative to the planet would be zero. Hence its velocity relative to the Sun would be the same as the velocity of the planet relative to the sun. Consequently any excess speed would represent an additional ΔV over and above the velocity of the planet. We can then observe that any excess speed would be the hyperbolic excess speed, the speed at infinity for a hyperbolic orbit. We can summarize this result by simply stating:

$$\Delta V = V_\infty \tag{4}$$

That is to say, the hyperbolic excess speed of a planetocentric escape orbit is just the ΔV available for insertion into the heliocentric transfer orbit. Likewise on arrival, the ΔV required to enter into the target planet's orbit is the hyperbolic excess speed of arrival of the vehicle with respect to the target planet. We can now treat this aspect of the flight in a more formal manner. The velocity of the vehicle with respect to the Sun is the sum of the velocity of the vehicle with respect to the planet plus the velocity of the planet relative to the Sun. We have then at point (1)

$$\begin{aligned} \vec{V}_{v/s} = \vec{V}_1 &= \vec{V}_{v/p} + \vec{V}_{p/s} \\ &= \vec{V}_{\infty_1} + \vec{V}_{c_1} \\ &= \Delta \vec{V}_1 + \vec{V}_{c_1} \end{aligned} \quad (5)$$

We can draw the vector diagram to illustrate the location of these vectors.



Here V_{c_1} is the planet's circular velocity, V_1 the transfer orbit velocity, ϕ_1, the flight path angle of the transfer orbit, ΔV_1 the hyperbolic excess speed or equivalently, the required ΔV to enter into the transfer orbit, and β_1, the angle V_∞ makes with the circular velocity of the planet. This angle may be used later. From the heliocentric orbit properties, we know V_1 and ϕ_1. We also know the planet velocity relative to the Sun, (circular speed). Therefor we can calculate the required ΔV_1 from the law of cosines:

$$\begin{aligned} \Delta V_1^2 &= V_1^2 + V_{c_1}^2 - 2 V_1 V_{c_1} \cos\phi_1 \\ &= V_{\infty_1}^2 \end{aligned} \quad (6)$$

Example: In our example the transfer orbit is parallel to the Earth's orbit so:

$$\Delta V_1 = V_1 - V_{c_1} = 1.1705 - 1 = 0.1705 \text{ AU/TU}$$

7.2 **Departure-Planet-Centered Orbits**

Before discussing the departure planet escape orbit, we should be careful to note that we must be consistent with units. If we use basic units (km, km/s, etc) then we need to be sure that we are using the correct gravitational parameter μ. If we persist on using canonic units, then we must convert to the canonic units of the planet of interest. This conversion is done by the use of the reference distance and the reference velocity. The reference distance is the radius of the planet, and the reference velocity is the circular speed of a satellite at the orbit radius of the planet, $V_{ref} = \sqrt{\dfrac{\mu_{planet}}{R_{planet}}}$. The reference time is the time it takes to move through 1 radian in the reference orbit. For the sun, the reference distance is the Earth's orbit radius, (1 AU) and the reference speed is the Earth mean orbital speed (1 EMOS) = 1 AU/TU. To convert from one system to the other we can perform the following operation when going to or from heliocentric to planetocentric orbits.

$$V \text{ DU/TU} = \frac{\text{heliocentric ref speed}}{\text{planetocentric ref speed}} \; x \; V \text{AU/TU} \qquad (7)$$

$$V \text{ DU/TU} = \frac{29.784852 \text{ km/s/AU/TU}}{7.9053661 \text{ km/s/DU/TU}} \; x \; V \text{ AU/TU} \quad \text{(e.g. for Earth)}$$

In our example we have,

$$V_\infty = \Delta V_1 = 0.1705 \text{ AU/TU} \; x \; \frac{29.784852}{7.9053661} = 0.6424 \text{ DU/TU}$$

We now know that we need a hyperbolic orbit with respect to the departure planet that has a hyperbolic excess velocity of V_∞. If we leave from a parking orbit (or for that matter leave from the surface of the Earth), we can compute the required burnout velocity using the energy equation. For the planet-centered hyperbolic escape orbit, the energy equation takes the form:

$$\frac{V_{bo}^2}{2} - \frac{\mu}{r_{bo}} = \frac{V_\infty^2}{2} \qquad \Rightarrow \qquad V_{bo}^2 = V_\infty^2 + \frac{2\mu}{r_{bo}} \qquad (8)$$

For our example, if the escape burn were done from a circular parking orbit of radius 1.05 DU, then the burn out velocity would be:

$$V_{bo} = \sqrt{0.6424^2 + \frac{2\,(1)}{1.05}} = 1.5223 \text{ DU/TU}$$

If we were to leave from a circular parking orbit at some specified flight path angle, then the ΔV_{bo} required would be given by the law of cosines:

$$\Delta V_{bo}^2 = V_{c_{bo}}^2 + V_{bo}^2 - 2 V_{c_{bo}} V_{bo} \cos\phi_{bo} \qquad (9)$$

which, for the case of a tangent burn becomes $\Delta V_{bo} = V_{bo} - V_{c_{bo}}$. In most cases, one would leave tangentially from the parking orbit since the required ΔV needed to achieve a given burnout velocity would be the smallest.

For our example problem, we will leave tangent to the circular parking orbit, we have

$$\Delta V_{bo} = V_{bo} - V_{c_{bo}} = 1.5223 - \sqrt{\frac{1}{1.05}} = 0.5464 \text{ DU/TU}$$

The planet escape orbit is going to be a hyperbolic orbit with a hyperbolic excess velocity, $V_\infty = \Delta V_1$. The properties of this escape orbit can be determined from the energy and angular momentum equations applied to the burnout conditions. The velocity at burnout can be related to the hyperbolic excess velocity from the expression given previously. These calculations lead to the following result. The semi-major axis of the escape orbit is determined from the energy equation:

$$En = \frac{V_{bo}^2}{2} - \frac{\mu}{r_{bo}} = \frac{V_\infty^2}{2} = \frac{\mu}{2a} \quad \Rightarrow \quad a = \frac{\mu}{V_\infty^2} \tag{10}$$

The eccentricity of the escape orbit is given by:

$$e^2 = \left(\frac{r_{bo} V_{bo}^2}{\mu} - 1\right)^2 \cos^2\phi_{bo} + \sin^2\phi_{bo}$$

or equivalently,

Eccentricity of Escape Orbit

$$e^2 = \left(\frac{r_{b0} V_\infty^2}{\mu} + 1\right)^2 \cos^2\phi_{bo} + \sin^2\phi_{bo} \tag{11}$$

For the special case where the escape burn is done tangentially and thus occurs at the periapsis of the escape orbit, we can determine the eccentricity from the expression for the periapsis, $r_p = a\,(1 - e)$, and using the previous expression for a,

Eccentricity of escape orbit if escape burn at periapsis

$$e = \frac{r_p}{a} + 1 = \frac{r_p V_\infty^2}{\mu} + 1 \tag{12}$$

For our example, since we are launching from perigee, we can use the short form for e,

$$e = 1 + \frac{1.05(0.6424)}{1} = 1.4333$$

Then we have, in either case,

$$\cos\nu_\infty = -\frac{1}{e} \qquad (13)$$

For our example,

$$\cos\nu_\infty = -\frac{1}{e} = -\frac{1}{1.4333} = -0.6977 \qquad \Rightarrow \qquad \nu_\infty = 134.24 \text{ deg}$$

The expression for the true anomaly at insertion into the escape orbit can be determined in a couple of ways:

$$r_{bo} = \frac{a(e^2 - 1)}{1 + e\cos\nu_{bo}} \qquad \Rightarrow \qquad \cos\nu_{bo} = \frac{1}{e}\left[\frac{a(e^2 - 1)}{r_{bo}} - 1\right]$$

or

$$Tan\,\nu_{bo} = \frac{\frac{r_{bo}V_{bo}^2}{\mu}\sin\phi_{bo}\cos\phi_{bo}}{\frac{r_{bo}V_{bo}^2}{\mu}\cos^2\phi_{bo} - 1} = \frac{\left(\frac{r_{bo}V_\infty^2}{\mu} + 2\right)\sin\phi_{bo}\cos\phi_{bo}}{\left(\frac{r_{bo}V_\infty^2}{\mu} + 2\right)\cos^2\phi_{bo} - 1} \qquad (14)$$

At periapsis insertion, $\nu_{bo} = \phi_{bo} = 0$.

The next calculation relating to the escape orbit is that relating to the "patch" conditions. In particular we need to know the angle β_1 between the hyperbolic excess velocity vector and the planet velocity vector with respect to the Sun. Assuming a circular planet orbit we have from the previous figure:

$$V_\infty \sin\beta_1 = V_1 \sin\phi_1 \qquad \text{and} \qquad V_1\cos\phi_1 = V_{c_1} + V_\infty\cos\beta_1$$

which leads to,

$$\sin\beta_1 = \frac{V_1}{V_\infty}\sin\phi_1 = \frac{V_1}{\Delta V_1}\sin\phi_1$$

or preferably,

$$\tan\beta_1 = \frac{V_1\sin\phi_1}{V_1\cos\phi_1 - V_{c_1}} \qquad (15)$$

The final calculation deals with the location of the escape burn. The angle will be called the launch angle θ_L, and is measured from the positive direction of the planet's heliocentric velocity, counter clockwise to the location on the circular parking orbit at which the escape burn takes place. From geometry,

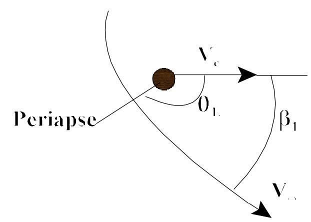
$$\theta_L = \nu_\infty + \beta_1 - \nu_{bo} \qquad (16)$$

For periapsis launch, $\nu_{bo} = 0$, and we have

$$\theta_L = \nu_\infty + \beta_1 \qquad \text{(periapsis launch)}$$

For our example problem, since β_1 and $\nu_{bo} = 0$,

$$\theta_L = \nu_\infty = 134.24 \text{ deg}$$



7.3 Arrival Planet-Centered Orbits

The "patch" conditions at the arrival planet are essentially the reverse of those at the departure planet. Here we compute the relative velocity to the planet, or the ΔV required for planet velocity match and that, as for departure, gives us V_∞ relative to the planet. This calculation is the same:

$$V_\infty^2 = V_2^2 + V_{c_2}^2 - 2V_2V_{c_2}\cos\phi_2 \qquad (17)$$

where V_2 = Heliocentric arrival velocity

V_{c_2} = Circular velocity of arrival planet with respect to the Sun

Again one must be careful to use consistent units. Here they are Sun canonic units.

For our example problem the arrival condition at Mars is determined from the angular momentum and energy of the heliocentric orbit.

$$\frac{V_2^2}{2} - \frac{\mu}{r_2} = \frac{V_1^2}{2} - \frac{\mu}{r_1} = En = \frac{V_2^2}{2} - \frac{1}{1.524} \qquad \Rightarrow \qquad V_2 = 0.8261 \text{ AU/TU}$$

$$h = r_p V_p = (1)(1.1705) = 1.524(0.8261)\cos\phi_2 \qquad \Rightarrow \qquad \cos\phi_2 = 0.9297$$

$$\phi_2 = 21.61 \text{ deg}$$

$$V_{c_2} = \sqrt{\frac{1}{1.524}} = 0.8100 \text{ AU/TU}$$

$$V_\infty^2 = 0.8261^2 + 0.8100^2 - 2(.8261)0.8100(0.9297) = 0.0943$$

$$V_\infty = 0.3071 \text{ AU/TU}$$

The conditions at the "patch" point allow us to determine the angle V_∞ makes with the arrival planet's heliocentric velocity vector (β_2). The calculations are the same as departure, and the picture is essentially the same. Designating the arrival parameters with a subscript 2, we have

$$\tan\beta_2 = \frac{V_2 \sin\phi_2}{V_2 \cos\phi - V_{c_2}} \qquad (18)$$

where:

ϕ_2 = flight path angle of heliocentric transfer orbit at arrival and V_2, and V_{c_2} are as defined previously.

For our example problem,

$$\tan\beta_2 = \frac{0.8261\sin(21.61°)}{0.8261\cos(21.61) - 0.8100} = -7.2511$$

$$\beta_2 = 97.85 \text{ deg}$$ 2nd quadrant (+/-)

Convert to Mars canonical units:

$$1 \text{ SU} = \sqrt{\frac{\mu_{mars}}{R_{mars}}} = \sqrt{\frac{4.305x10^4}{3380}} = 3.5688 \text{ km/s}$$

$$V_\infty = 0.3071 \text{ AU/TU } x \ \frac{29.784852}{3.5688} = 2.5630 \text{DU/TU}_{\text{mars}}$$

It is assumed that the periapsis height above the planet is given, r_p is known. Then we can compute the planetocentric orbit parameters from the following equations, the same as used for the escape orbit previously.

$$En = \frac{V_\infty^2}{2} = \frac{\mu}{2a} \qquad \Rightarrow \qquad a = \frac{\mu}{V_\infty^2} \tag{19}$$

$$e = 1 + \frac{r_p V_\infty^2}{\mu} \tag{20}$$

$$\cos \nu_\infty = -\frac{1}{e} \tag{21}$$

We can define a distance d, which is the perpendicular distance of the vector V_∞ from the planet. This distance is a function of the periapsis distance and is determined from angular momentum and energy considerations. From the definition of d, we have the following expression for the angular momentum:

$$h = V_\infty d \tag{22}$$

and of course the familiar expression for energy

$$\frac{V_p^2}{2} - \frac{\mu}{r_p} = \frac{V_\infty^2}{2} \tag{23}$$

Now $h = V_\infty d = r_p V_p \qquad \Rightarrow \qquad V_p = \frac{d}{r_p} V_\infty$ If we substitute this expression into the energy equation and solve for d, we get an expression for d that contains only the periapsis distance and V_∞.

$$d = r_p \sqrt{1 + \frac{2\mu_{planet}}{r_p V_\infty^2}} \qquad (24)$$

To get this distance, we must plan to arrive in front of or behind the planet by some distance x. This distance is given by:

$$x = \frac{d}{\sin\beta_2} \qquad (25)$$

Note that for a Hohmann transfer this last analysis collapses. For a Hohmann, you need to target a smaller or larger radius, by the amount d. For all practical purposes in mission analysis, we can assume the same Hohmann ΔVs and just say that it gets there at the correct distance from the planet with a relative velocity parallel to the planet orbit velocity with $\beta_2 = 0 \; or \; \pi$ (0 for planet radii less than departure planet radius, and π for arrival planet radii greater than departure planet radius).

The hyperbolic entry orbit to the planet will strike the planet if r_p is less than the planet radius. The corresponding value of d is called the ***collision radius*** of the planet. Approaches within this radius will collide with the planet. If r_p is greater than that value which leads to collision, the satellite will miss the planet, and continue on a hyperbolic orbit until it leaves the planet. Since there is no energy lost, (assuming there is no atmosphere to cause drag), the satellite leaves with the same magnitude of excess velocity as it had when it arrived.

At any given position in the hyperbolic orbit, we can calculate the true anomaly and flight path angle in the orbit. These are calculated from the now familiar equations:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{V_\infty^2}{2} \quad \Rightarrow \quad V = \sqrt{V_\infty^2 + \frac{2\mu}{r}} \qquad (26)$$

$$\cos\phi = \frac{h}{rV} = \frac{V_\infty d}{rV} \qquad (27)$$

$$Tan\,\nu = \frac{\frac{rV^2}{\mu}\sin\phi\cos\phi}{\frac{rV^2}{\mu}\cos^2\phi - 1} = \frac{\left(\frac{rV_\infty^2}{\mu} + 2\right)\sin\phi\cos\phi}{\left(\frac{rV_\infty^2}{\mu} + 2\right)\cos^2\phi - 1} \qquad (28)$$

Of course if we elect to put the vehicle in some orbit at radius r, then we need to do a capture burn. Assuming the planetocentric parking orbit is a circular one, the required ΔV is given by

$$\Delta V_{capture}^2 = V^2 + V_c^2 - 2 V V_c \cos\phi \tag{29}$$

Where V_c is the circular velocity of the desired parking orbit at radius r, $V_c = \sqrt{\frac{\mu}{r}}$

In most cases an attempt will be made to enter the parking orbit tangentially at the periapsis, so that the following relations hold:

$$\phi = \nu = 0 \qquad \Delta V_{capture} = V_c - V_p \qquad \text{(periapsis capture)}$$

For our example problem if we enter a circular orbit at 1.1 Mars radius at perimars, we have the following calculations: (using Mars canonic units)

$$V_p = \sqrt{V_\infty^2 + 2\frac{\mu}{r_p}} = \sqrt{2.5630^2 + \frac{2}{1.1}} = 2.8961 \text{ DU/TU}_{\text{mars}}$$

$$\Delta V_p = \Delta V_{capture} = V_c - V_p = \sqrt{\frac{1}{1.1}} - 2.8961 = -1.9462 \text{ DU/TU}_{\text{mars}}$$

(-) Indicates slowing down

$$e = 1 + \frac{r_p V_\infty^2}{\mu} = 1 + \frac{1.1(2.5630)^2}{1} = 8.2259$$

$$\cos\nu_\infty = -\frac{1}{e} = -\frac{1}{8.2259} \qquad \Rightarrow \qquad \nu_\infty = 96.98 \text{ deg}$$

The location of the capture is determined from a similar equation for determining the launch burnout location. Hence we have:

$$\theta_{capture} = \beta_2 + \nu_\infty - \nu_{capture} \tag{30}$$

$$\theta_{capture} = 97.85 + 96.98 - 0 = 194.83 \text{ deg}$$

Flyby

If the vehicle is not captured, then it will fly by and leave at the same relative velocity as it approached (V_∞). In this case we must do additional calculations to determine the heliocentric orbit on leaving. First we need to determine the turning angle, or the angle between the approach asymptote and the departure asymptote. This angle is given by:

$$\sin\delta/2 = \frac{1}{e} \tag{31}$$

where δ is the turning angle.

For our example, $\sin\frac{\delta}{2} = \frac{1}{8.2259} \quad \Rightarrow \quad \delta = 13.96 \text{ deg}$

Therefore the angle the departure V_∞ makes with the planet heliocentric velocity vector, β_3, is given by;

$$\beta_3 = \beta_2 \pm \delta$$

+ for flight under the planet

- for flight over the planet

For our example: $\beta_3 = 97.85 - 13.96 = 83.89 \text{ deg}$.

Then the "patch" conditions lead to the following equations. These are similar to those for departure from the first planet, and entry to the second planet. However different quantities are known so the equations appear slightly different. It is still the application of the law of cosines. We will designate the conditions at departure from the flyby as 3. Note that the planet orbit's radius $r_2 = r_3$, and $V_{c_3} = V_{c_2}$.

$$V_3^2 = V_\infty^2 + V_{c_3}^2 + 2V_\infty V_{c_3}\cos\beta_3 \tag{32}$$

$$\tan\phi_3 = \frac{V_\infty \sin\beta_3}{V_\infty\cos\beta_3 + V_{c_3}} \tag{33}$$

For our example: (these are back in Sun canonic units)

$$V_3^2 = 0.3071^2 + 0.8100^2 + 2\,0.3071(0.8100)\cos(83.89°)$$
$$V_3 = 0.8963 \text{ AU/TU}$$

$$\tan\phi_3 = \frac{0.3071\sin(83.89)}{0.3071\cos(83.89) + 0.8100} = 0.3623$$

$\phi_3 = 19.92 \text{ deg}$ (First quadrant +/+)

7.4 Heliocentric Orbit After Flyby

From the information calculated on leaving the planet, we can determine the characteristics of the new heliocentric orbit after the flyby. To do this, we need the energy and angular momentum.

$$\frac{V_3^2}{2} - \frac{\mu}{r_3} = -\frac{\mu}{2a} = En \quad (34)$$

$$h = r_3 V_3 \cos^{\prime}\phi_3 \quad (35)$$

$$\tan\nu_3 = \frac{\frac{r_3 V_3^2}{\mu}\sin\phi_3\cos\phi_3}{\frac{r_3 V_3^2}{\mu}\cos^2\phi_3 - 1} \quad (36)$$

This information is sufficient to determine all properties of the heliocentric orbit. We could continue to flyby another planet by repeating the calculations for points 2 and 3. Notice that the radius of the sphere of influence does not enter into these mission analysis calculations. The original heliocentric orbit and the new heliocentric orbit terminate and initiate at the orbit radius of the flyby planet. ($r_2 = r_3$). Hence the heliocentric orbit is discontinuous in energy, angular momentum, and consequently, flight path angle and velocity at the planet orbit radius. Over flights tend to increase the energy while under flights may decrease the energy.

Example problem continued:

$$\frac{V_3^2}{2} - \frac{\mu}{r_3} = \frac{0.8963^2}{2} - \frac{1}{1.524} = -0.2545 \ \frac{\text{AU}^2}{\text{TU}^2}$$

$$h = 1.524(0.8968)\cos(19.92°) = 1.2842 \ \text{AU}^2/\text{TU}$$

These are the new heliocentric orbit properties.

Energy Gained During Flyby

The energy gained during a flyby (over the planet) can be established in a manner that gives more insight to the problem. We will continue the notation from above and consider item (2) to be the approach conditions, and item (3) to be the departure conditions. In addition, our approximation includes the assumption that the sphere of influence relative to the Sun is zero, so that the distance from the Sun at arrival and departure is the same. Under these assumptions, the energy gained during a flyby is given by:

$$\Delta En = \left(\frac{V_3^2}{2} - \frac{\mu}{r} \right) - \left(\frac{V_2^2}{2} - \frac{\mu}{r} \right) = \frac{1}{2}\left(V_3^2 - V_2^2 \right) \tag{37}$$

We can represent the approach and departure vectors as:

$$\vec{V}_3 = \vec{V}_p + V_\infty \hat{O} \qquad\qquad \vec{V}_2 = V_p + V_\infty \hat{I} \tag{38}$$

where $\vec{V}_p$ = planet velocity

V_∞ = hyperbolic excess speed at approach and departure

$\hat{O}$ = unit vector along the departure asymptote

$\hat{I}$ = unit vector along the approach asymptote

If we take the scalar of $\vec{V}_2$ and $\vec{V}_3$ with themselves to get their squares, we can substitute them into Eq. (37) to get:

$$\Delta En = \frac{1}{2}\left[V_p^2 + V_\infty^2 + 2 V_p V_\infty \hat{P}\cdot\hat{O} - V_p^2 - V_\infty^2 - 2 V_p V_\infty \hat{P}\cdot\hat{I} \right]$$

where $\hat{P}$ is a unit vector along the planet velocity direction, or,

$$\boxed{\Delta En = V_p V_\infty \hat{P}\cdot(\hat{O} - \hat{I}) = V_p V_\infty (\cos\beta_3 - \cos\beta_2)} \tag{39}$$

From Eq. (39) we can make the following observations:

1) For a given direction of arrival, $\hat{I}$ we can maximize the increase in energy when $\hat{P}\cdot\hat{O}$ is maximum, or when the exit or departure velocity is parallel to the planet velocity. In this case the heliocentric orbit after leaving the planet would be tangent to the planets orbit velocity (Homann-like).

2) For a given turning angle, δ, $(\hat{O} - \hat{I} = \text{const})$, the maximum increase in energy occurs when $\hat{P}\cdot(\hat{O} - \hat{I})$ is a maximum which occurs when the $(\hat{O} - \hat{I})$ is parallel to the planet velocity. Because of symmetry, this occurs when the major axis of the hyperbola is aligned with the planet velocity.