9. The Orbit in Space - 3D Orbit

Up until now, (with the exception of the plane change maneuver), all our discussion have been restricted to the orbit plan or to a two dimensional world. Now, however, we need to describe the orbit in a three dimensional space. There are many ways to describe the 3D orbit of which two will be discussed here, classical orbital elements, or with position and velocity vectors. In either case we must introduce some coordinate systems in which to represent the various vectors of interests.

Coordinate Systems

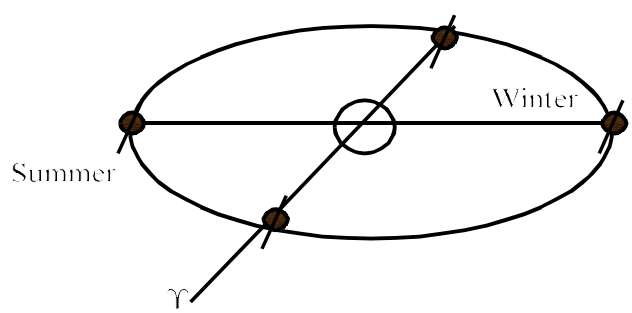
All coordinate systems have a reference plane (usually the x-y plane) and a reference direction in that plane (usually the x axis). We wish to study the motion of artificial satellites about the Earth and real and artificial satellites about the Sun and possibly other planets. We will restrict ourselves to Earth-centered (geocentric) and Sun-centered (heliocentric) satellites. For each of these cases, the origin of the coordinate system of interest is at the center of the Earth or Sun. ***The reference plane*** for each of these coordinate systems is as follows:

1. Heliocentric Orbits - The reference plane for heliocentric orbits is the plane of the Earth's orbit about the Sun (or the apparent path of the Sun about the Earth from our point of view). This plane is called the ***plane of the ecliptic***, and for all practical purposes is considered an inertial plane.

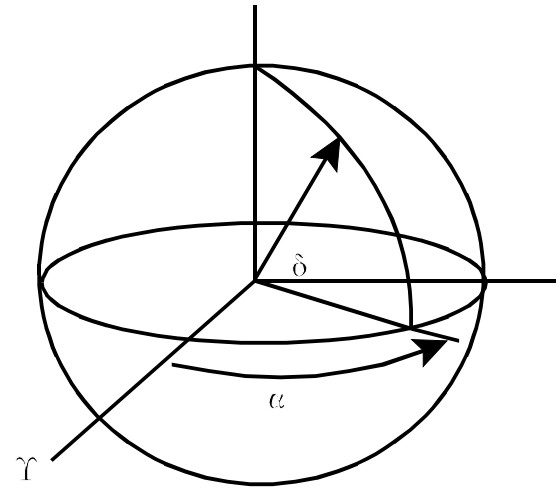
2. Geocentric Orbits - The reference plane for geocentric orbits is the equatorial plane of the Earth. Since the Earth's spin axis is tipped approximately 23.5 degrees from the normal to its orbit, the two reference planes do not coincide.

The two reference planes intersect along a line that is obviously common to both planes. When the Sun (as observed from the Earth) is along one of these lines, the day and night are of equal length. Hence these points are called equinoxes and are designated the spring (or vernal) and autumnal equinox. If we look from the Earth to the Sun at the vernal equinox we are looking in the ***common reference direction*** for both the heliocentric and geocentric coordinate systems.

Definition - Vernal Equinox - The Vernal Equinox (also called the "First Point of Aires") is designated by ♈ and is a "fixed" direction in space used as the reference direction. It is the direction from the Earth to the Sun when the Sun appears to pass from the Southern hemisphere to the Northern hemisphere.



Definition - Celestial Sphere - The celestial sphere is an imaginary sphere of infinite radius that is centered at the origin of the coordinate system of interest. The position of the satellites and stars are projected onto the celestial sphere and can be located by specifying two angles: one measured in the reference plane from the reference direction, and the other in a plane perpendicular to the reference plane. The two angles are designated as:



α ≡ right ascencion

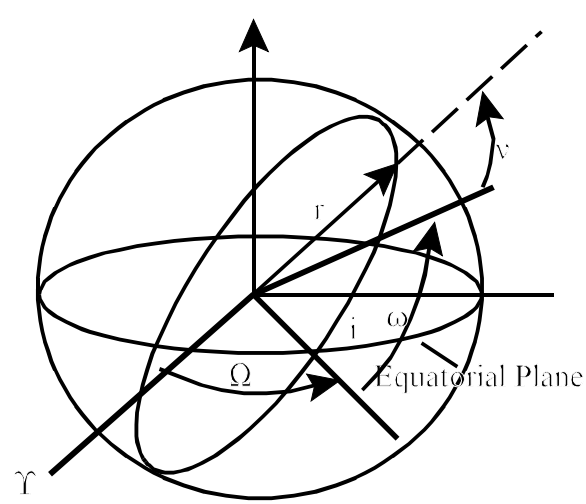
δ ≡ declination

This coordinate system is often used by astronomers to locate stars in the heavens.

Orbital Elements

Recall that we started our two-body problem with a vector, second order, ordinary differential equation that requires six constants of integration to obtain a complete solution to the problem. We extracted these six constants in our previous work. By combining these constants in certain ways, we can arrive at a different set of six independent constants. There are many ways to do these combinations and permutations and hence there are a large number of different groups of six constants. Whatever the final selection is, these six constants are called the orbital elements. Here we will look at the "classical" set of orbital elements.

The classic orbital elements include two elements to locate the plane of the orbit in space, two to describe the size and shape of the orbit, one to orient the orbit in the plane, and the last to locate the satellite in the orbit. ***The elements*** are:



a	Semi-major axis
e	Orbit eccentricity
i	the orbit inclination
Ω	the right ascension of the ascending node (Or longitude of the ascending node)
ω	argument of periapsis
τ	time of periapsis passage

The orbit plane and the reference plane (equatorial plane) intersect in a line called the "line of nodes". The end of this line where the satellite moves from the southern hemisphere to the northern hemisphere is called the ascending node. It takes two angles to orient the orbit plane in space, the angle between the vernal equinox and the ascending node, Ω, called the longitude of the ascending node (or right ascension of the ascending node), and the angle between the normal to the orbit plane (along the angular momentum vector), and the normal to the reference plane, the z or north-pointing axis, called the inclination angle, *i*. As will be shown later, these two angles are directly related to the angular momentum vector as might be expected. We should also note that we define direct and retrograde orbits in the following manner:

$0 \le i < 90$ deg	Direct orbit
$90 < i \le 180$deg	Retrograde orbit
$i = 90$ deg	Polar orbit

The energy of the orbit gives us the size of the orbit, or the semi-major axis, *a*, and the magnitude of the angular momentum (combined with the energy) give us the shape or eccentricity of the orbit. Finally the orbit is oriented in the plane by defining the angle, ω, between the line of the ascending node, and the line passing through the periapsis of the orbit. This angle is called the argument of periapsis and is measured in the plane of the orbit.

The final constant is the time of periapsis passage, τ, that allows us to determine where the satellite is in orbit at some given time, t.

These six elements are called the "classic" orbital elements and fully describe the orbit and the position of the satellite in orbit. Hence if we know the elements and time, a, e, i, Ω, ω, τ, and t, we can locate the orbit and the satellite in space. These elements are convenient to use because they bring the physical picture to mind and allow one to visualize the orbit and its orientation. However, often other "elements" are used to describe orbits. Rather than use t and τ, it is convenient to us the mean anomaly M or more properly, the mean anomaly at some specified time or epoch, M_0. Hence the orbital elements are a, e, i, Ω, ω, and M_0. (Note that M_0 may be considered an element since it is a constant). In either case, in order to determine the satellite's position, we are required to solve Kepler's equation in order to determine E, and subsequently, ν, the true anomaly. It is convenient to designate the true anomaly, or more properly, the true anomaly at some specified time or epoch as an "element." We can completely describe the orbit and satellite position is space if we know the "elements" a, e, i, Ω, ω, and ν_0. In this form, we do not have to solve Kepler's equation and thus this form will be used extensively here for convenience. However, you should be aware that the true anomaly, ν, is not an orbital element while ν_0 may be considered an element. Use of the true anomaly as an orbital element avoids the necessity of solving the Kepler equation. If the classic orbital elements and time are given, ν and ν_0 would be calculated from the Kepler and related equations given a, e, τ, and t.

Alternate Orbital Elements

A cursory glance at the classical orbital elements will reveal that under certain circumstances, such as an equatorial orbit or a circular orbit (or both), that some elements are

undefined. Motivated by this occurrence, alternative elements have been defined. Again, these are combinations of existing elements.

$u_0 = \omega + \nu_0$ argument of latitude at epoch

This also defines the time varying version of this element

$u(t) = \omega + \nu(t)$ argument of latitude

We also have

$\varpi = \Omega + \omega$ longitude of periapsis

$\ell_0 = \varpi + \nu_0$ true longitude at epoch
$\ell(t) = \varpi + \nu(t)$ true longitude

ν_0 true anomaly at epoch
$\nu(t)$ true anomaly

The Three Dimensional Problems

There are two problems of interest that we can study in this three dimensional environment

1) Given: the orbital elements a, e, i, Ω, ω, τ, and the time t (or another set of elements),
Find: the position and velocity vectors, $\vec{r}$, $\vec{V}$.

2) Given: the position and velocity vectors, $\vec{r}$, $\vec{V}$,
Find: the orbital elements a, e, i, Ω, ω, τ.

Problem (2) is called the orbit determination problem.

<u>Problem (1) Given the orbital elements - find the position and velocity vectors</u>

The problem requires that we find the position and velocity vectors in an inertial coordinate system. The approach is to find the representation of the position and velocity vectors in some convenient coordinate system and then find a transformation that allows us to find their representation in the desired “inertial” system. This idea of transferring the representation of vectors from one system to another is an extremely useful concept and is used in many other astromechanics problems as well as in many other disciplines.. We relate these two representations in the two different coordinate systems to each other by use of what are called transformation matrices.

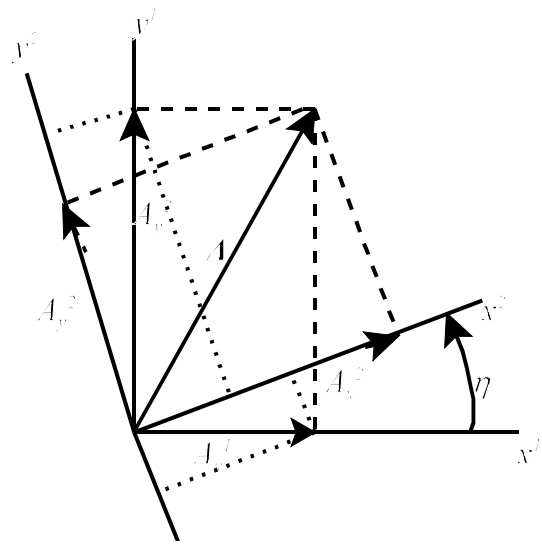
Transformation Matrices

In order to develop the concept of a transformation matrix we will look at the representation of a generic vector, $\vec{A}$, in two different coordinate systems labeled (1) and (2). In addition, coordinate system (2) is rotated about the z axis an amount η with respect to system (1). The vector representation in each coordinate system is given by:

$$\vec{A}^1 = A_x \hat{i}^2 + A_y^1 \hat{j}^1 + A_z^1 \hat{k}^1$$

and

$$\vec{A}^2 = A_x^2 \hat{i}^2 + A_y^2 \hat{j}^2 + A_z^2 \hat{k}^2$$



where the superscript indicates the coordinate system in which the quantity is represented. We can relate the components of the vector represented in system (2) to the components of the same vector written in system (1) by observing the figure and recalling that the component of any vector along a given line is found by dropping a perpendicular to that line from the tip of the vector. In any case we have:

$$\begin{aligned} A_x^2 &= A_x^1 \cos\eta + A_y^1 \sin\eta \\ A_y^2 &= -A_x^1 \sin\eta + A_y^1 \cos\eta \\ A_z^2 &= A_z^1 \end{aligned} \tag{1}$$

These equations can be more presented more compactly using a matrix formulation:

$$\begin{Bmatrix} A_x^2 \\ A_y^2 \\ A_z^2 \end{Bmatrix} = \begin{bmatrix} \cos\eta & \sin\eta & 0 \\ -\sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_x^1 \\ A_y^1 \\ A_z^1 \end{Bmatrix} \tag{2}$$

or

$$\vec{A}^2 = \left[\eta_z\right]\vec{A}^1 \tag{3}$$

where $\left[\eta_z\right]$ is the transformation matrix from system (1) to system (2). Since in this case axes

system (2) is rotated an amount η from system (1), $\left[\eta_z\right]$ is called the transformation matrix for a z rotation or the ***z rotation transformation matrix***.

In a similar manner we can develop a transformation matrix for a single rotation about the x, y, and z axes. The results of such activity are given by:

$$\left[\eta_x\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & \sin\eta \\ 0 & -\sin\eta & \cos\eta \end{bmatrix}$$

$$\left[\eta_y\right] = \begin{bmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{bmatrix} \tag{4}$$

$$\left[\eta_z\right] = \begin{bmatrix} \cos\eta & \sin\eta & 0 \\ -\sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrices presented in Eq. (4) are the fundamental rotation matrices and represent the transformation matrices that allow us to transform the representation of a given vector in system (1) to the representation of the same vector in system (2) for a particular single rotation about the specified axis.

The orientation of one axes system with respect to another axes system can always be defined as a sequence of rotations about these individual axes. In general it takes three successive rotations to go from one system to an arbitrary orientation of the new system. However in some cases it may take less or for convenience, more than three. The sequence of rotations is important, and must be specified. Generally we label the x axis as the “1" axis, the y axis as the “2" axis and the z axis as the “3" axis. The sequence of rotation is then designated as the 321 or 313 sequence. There are 12 possible 3 rotation sequences. Of interest to us is the transformation from some convenient axes system in which it is easy to represent the position and velocity to the standard inertial system where the vernal equinox serves as the “x” axis and the north point axis as the “z” axis.

The Local Coordinate System

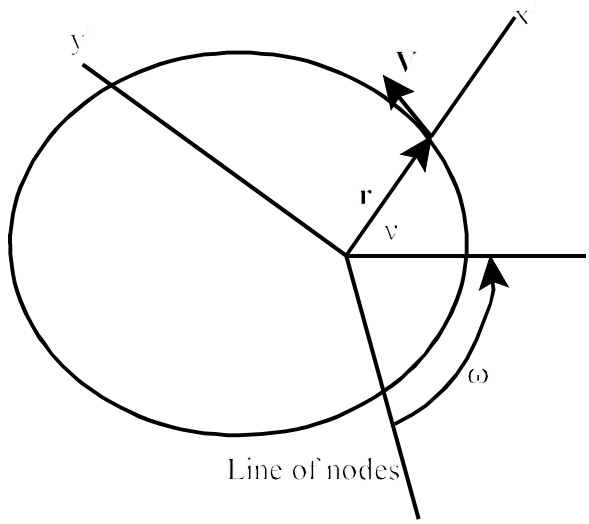
For convenience we will define the “local” coordinate system in the following way: The

origin of the system is geocentric (heliocentric or planetocentric) with the x axis point out the radius vector. The y axis is at right angles to the x axis and lies in the plane of the orbit. The z axis is normal to the plane of the orbit and is aligned with the angular momentum vector. The three axes form a right handed set. The representation of the position vector in such an axis system is extremely simple:

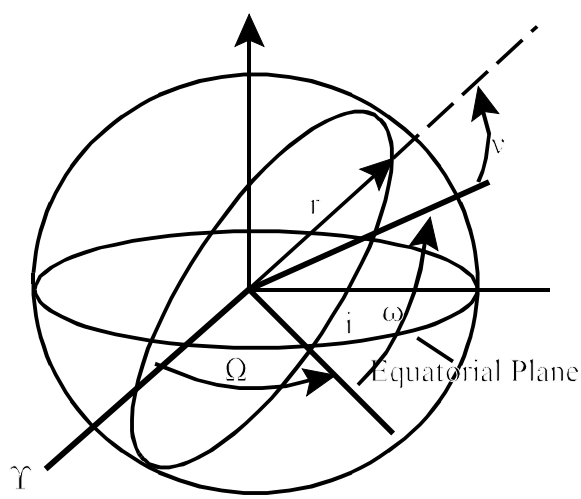
$$\vec{r} = \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}^{\ell} \qquad (5)$$

and the velocity is simply:

$$\vec{V} = \begin{Bmatrix} \dot{r} \\ r\,\dot{\nu} \\ 0 \end{Bmatrix} \qquad (6)$$



We now seek the sequence of rotations that it would take to align the "local" axes system with the standard inertial system. From the two figures we can establish that the following sequence of rotations will align the local axes system with the classical inertial system:



1) rotate $[-(\omega + \nu)]$ about the z^{ℓ} axis to x'y'z' set (x axis now aligned with line of nodes)

2) rotate (- *i)* about the x' axis to the x"y"z" set (x axis aligned with line of nodes, and the z axis pointing north)

3) rotate (- Ω) about the z" axis to the inertial ♈ YZ axes.

The transformation matrix to take the simple representations of the position and velocity vectors given by Eqs. (5) and (6) into a representation of the same vectors in the inertial system are obtained by carrying out the transformations one rotation at a time. We also note the following (well known) identities:

$$\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta \end{aligned} \qquad (7)$$

The transformation matrix has the form:

$$\vec{r}^{\,I} = \left[-\Omega_z\right]\left[-i_x\right]\left[-(\omega+\nu)_z\right]\vec{r}^{\,\ell} \tag{8}$$

with a similar form for the velocity.

We can write out Eq. (8) in more detail, and can multiply the matrices together to get the final transformation matrix.

$$\begin{Bmatrix} X \\ Y \\ X \end{Bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos(\omega+\nu) & -\sin(\omega+\nu) & 0 \\ \sin(\omega+\nu) & \cos(\omega+\nu) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

$$= T_{\ell 2I}\vec{r}^{\,\ell}$$

where $T_{\ell 2I}$ is the transformation matrix given by:

$$t_{\ell 2I} = \begin{bmatrix} \cos\Omega\cos(\omega+\nu) - \sin\Omega\sin(\omega+\nu)\cos i & -\cos\Omega\sin(\omega+\nu) - \sin\Omega\cos(\omega+\nu)\cos i & \sin\Omega\sin i \\ \sin\Omega\cos(\omega+\nu) + \cos\Omega\sin(\omega+\nu)\cos i & -\sin\Omega\sin(\omega+\nu) + \cos\Omega\cos(\omega+\nu)\cos i & -\cos\Omega\sin i \\ \sin(\omega+\nu)\sin i & \cos(\omega+\nu)\sin i & \cos i \end{bmatrix} \tag{10}$$

Equation (10) provides the complete transformation matrix for transferring vectors represented in the local system to the same vectors represented in the inertial system. In order to get our desired results, we must multiply the matrix by the position and velocity vectors. For the position vector the result is just *r* times the first column of the transformation matrix:

$$\vec{r}^{\,I} = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{Bmatrix} r\left[\cos\Omega\cos(\omega+\nu) - \sin\Omega\sin(\omega+\nu)\cos i\right] \\ r\left[\sin\Omega\cos(\omega+\nu) + \cos\Omega\sin(\omega+\nu)\cos i\right] \\ r\left[\sin(\omega+\nu)\sin i\right] \end{Bmatrix} \tag{11}$$

The velocity vector in classical inertial coordinates is

$$\begin{Bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{Bmatrix} = \begin{Bmatrix} \dot{r}\left[\cos\Omega\cos(\omega+\nu) - \sin\Omega\sin(\omega+\nu)\cos i\right] + r\dot{\nu}\left[-\cos\Omega\sin(\omega+\nu) - \sin\Omega\cos(\omega+\nu)\cos i\right] \\ \dot{r}\left[\sin\Omega\cos(\omega+\nu) + \cos\Omega\sin(\omega+\nu)\cos i\right] + r\dot{\nu}\left[-\sin\Omega\cos(\omega+\nu) + \cos\Omega\cos(\omega+\nu)\cos i\right] \\ \dot{r}\left[\sin(\omega+\nu)\sin i\right] + r\dot{\nu}\left[\cos(\omega+\nu)\sin i\right] \end{Bmatrix} \tag{12}$$

where from previous work we have:

$$\dot{r} = \frac{\mu}{h} e \sin\nu$$
$$r\dot{\nu} = \frac{h}{r} = \frac{\mu}{h}\left(1 + e\cos\nu\right) \tag{13}$$

Hence given the orbital elements and time, a, e, i, Ω, ω, τ, and t, we can use Kepler's equation to solve for the true anomaly, ν, and use Eqs. (11) and (12) to obtain the position and velocity vectors $\vec{r}$ and $\vec{V}$ represented in the classical inertial system.

Perifocal Coordinate System

At this point we can introduce the perifocal coordinate system. In the two-body problem this is also an inertial coordinate system. Its origin is at the focus (geocentric or heliocentric) of the orbit with the x axis pointing in the direction of the periapsis, the y axis in the orbit plane, and the z axis aligned with the angular momentum vector, the three axes forming a right-hand set. It should be clear from the figure above that the transformation from the local coordinate system to the perifocal coordinate system is obtained by a simple rotation of $-\nu$ about the z axis. Therefore the transformation matrix from local to perifocal axes is given by:

$$T_{\ell tp} = \left[-\nu_z\right] = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Then the position and velocity vectors in the perifocal system can be determined to be:

For the position vector we have:

$$\left\{\vec{r}\right\}^p = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}^\ell = \begin{Bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{Bmatrix}^p \tag{15}$$

and for velocity we have:

$$\left\{\vec{V}\right\}^p = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\mu}{h} e\sin\nu \\ \frac{\mu}{h}(1 + e\cos\nu) \\ 0 \end{Bmatrix}^l = \begin{Bmatrix} -\frac{\mu}{h}\sin\nu \\ \frac{\mu}{h}(e + \cos\nu) \\ 0 \end{Bmatrix}^p \tag{16}$$

Equations (15) and (16) give the position and velocity in the peri-focal coordinate system in terms of the true anomaly, ν.

It is useful to have these same components of position and velocity in terms of the eccentric anomaly. In order to do this we need to go back to the basic definitions and geometry that we developed for introducing the eccentric anomaly. Eventually we will need an expression for the time derivative of the eccentric anomaly. We will develop that next. First recall that the magnitude of the position vector $\vec{r}$ is given by

$$r = a(1 - e\cos E) \tag{17}$$

From Kepler's equation we have:

$$M = n(t-\tau) = \sqrt{\frac{\mu}{a^3}}(t-\tau) = E - e\sin E \tag{18}$$

Taking the derivative with respect to time we get:

$$\sqrt{\frac{\mu}{a^3}} = (1 - e\cos E)\dot{E} = \frac{r}{a}\dot{E} \qquad \Rightarrow \qquad \dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \tag{19}$$

Then we can determine $\dot{r}$.

$$\dot{r} = ae\sin E\,\dot{E} = ae\sin E\left(\frac{1}{2}\sqrt{\frac{\mu}{a}}\right) = \frac{\sqrt{\mu a}\sin E}{r} \tag{20}$$

In the ***local system*** we have:

$$\vec{r} = \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}^{\ell} \qquad \vec{V} = \begin{Bmatrix} \dfrac{\sqrt{\mu a}\sin E}{r} \\ \dfrac{h}{r} \\ 0 \end{Bmatrix}^{\ell} \qquad \text{where} \quad r = a(1 - e\cos E) \tag{21}$$

To determine the components of the position and velocity vectors in the perifocal system we can go directly to the components already determined in the eccentric anomaly development. The position vector is given (from the geometry and definitions) and the velocity vector is obtained by taking the derivative of the components and using the expression developed previously for the time derivative of E.

In the ***perifocal system*** we have:

$$\vec{r} = \begin{Bmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\,\sin E \\ 0 \end{Bmatrix}^p \qquad \vec{V} = \begin{Bmatrix} -\dfrac{\sqrt{\mu a}\,\sin E}{r} \\ \dfrac{h}{r}\cos E \\ 0 \end{Bmatrix}^p \tag{22}$$

Finally we can develop the transformation matrix between the perifocal system and the classical inertial system. It is clear that if the true anomaly were zero, then the local an perifocal system would be aligned. Hence the transformation matrix from the perifocal system to the classical inertial system is simply the transformation matrix from the local system to the perifocal system with the true anomaly set to zero:

$$t_{p2I} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega) + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix} \tag{23}$$

<u>Problem 2 - Given the Position and Velocity Vectors, Determine the Orbital Elements - The Orbit Determination Problem</u>

The orbit determination problem is: Given the position and velocity vectors, $\vec{r}$ and $\vec{V}$ and time, *t*, determine the orbital elements, a, e, i, Ω, ω, and τ (or the true anomaly ν). Again, as in the first problem, there are many different approaches to solving this problem. One of the main difficulties with this problem is to be sure the angle that you get for Ω, ω, and ν is in the right quadrant.

Once we are given the position and velocity vector, $\vec{r}$ and $\vec{V}$ we can easily determine their magnitudes, *r*, and *V*. From these values we can determine the energy and hence the semi-major axis, and the angular momentum and hence the orbit eccentricity. The various angle orbital elements are determined by considering the angles between two vectors. For example if we have two vectors, $\vec{A}$ and $\vec{B}$, then the angle between these two vectors can be determined from the scalar product: $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\cos\alpha$, where α is the angle between the two vectors. For the orbit determination problem we will be interested in four vectors:

1. The position vector, $\vec{r}$
2. The angular momentum vector, $\vec{h} = \vec{r} \times \vec{V}$
3. The node vector, $\vec{n} = \hat{k}^I \times \vec{h}$
4. The eccentricity vector or the Laplace vector that has the magnitude of the eccentricity

and is directed in the direction of the periapsis, $\vec{e}$

We need to develop the expression for this eccentricity vector. This development is somewhat abstract in that there is little motivation to carry out the steps as indicated. One should remember that orbital mechanics has been around since the days of Newton so there has been a lot of "playing" with the equations. When such activity lead to a result, the usual description of the development is "elegant." What follows is an elegant derivation of the eccentricity vector.
One of the tools we will use is the vector triple product identity or the bac - cab identity:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{B}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{24}$$

We will start with the basic equation of motion:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \tag{25}$$

We can now take the cross product with the angular momentum vector:

$$\begin{aligned} \ddot{\vec{r}} \times \vec{h} &= -\frac{\mu}{r^3}\vec{r} \times (\vec{r} \times \vec{V}) = -\frac{\mu}{r^3}\left[\vec{r}(\vec{r}\cdot\vec{V}) - \vec{V}(\vec{r}\cdot\vec{r})\right] \\ &= -\frac{\mu}{r^3}\left[r\dot{r}\vec{r} - r^2\vec{V}\right] = \frac{\mu}{r}\vec{V} - \frac{\mu\dot{r}}{r^2}\vec{r} \end{aligned} \tag{26}$$

where we have made use of the fact that for any vector, $\vec{A}\cdot\dot{\vec{A}} = A\dot{A}$. The left and right hand sides of Eq. (26) are perfect differentials, so we have:

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \tag{27}$$

Or integrating:

$$\dot{\vec{r}} \times \vec{h} = \mu\frac{\vec{r}}{r} + \vec{B} \tag{28}$$

We now take the scalar product of Eq. (28) with $\vec{r}$ to get:

$$\vec{r}\cdot\left(\dot{\vec{r}} \times \vec{h}\right) = \mu\frac{\vec{r}}{r}\cdot\vec{r} + \vec{B}\cdot\vec{r} \tag{29}$$

If we keep the order of the terms the same, we can interchange the dot and cross in the first term and we get:

$$h^2 = \left(\vec{r} \times \dot{\vec{r}}\right)\cdot\vec{h} = \mu r + rB\cos\psi \tag{30}$$

We can solve Eq. (30) for the radius, r

$$r = \frac{h^2/\mu}{1 + \frac{B}{\mu}\cos\psi} = \frac{h^2/\mu}{1 + e\cos\nu} \tag{31}$$

From Eqs. (29-31) we see that the angle between $\vec{B}$ and $\vec{r}$ is ν, the true anomaly, and that the magnitude of the vector $\vec{B}$, $B = \mu e$. Consequently, $\vec{B}$ is a vector that has a magnitude equal to the eccentricity and point in the direction of the periapsis. Hence we can define the eccentricity vector from $\vec{B} = \mu\vec{e}$. Then from Eq. (28) we have:

$$\mu\vec{e} = \dot{\vec{r}} \times \left(\vec{r} \times \dot{\vec{r}}\right) - \mu\frac{\vec{r}}{r} = \left[\vec{r}\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right) - \dot{\vec{r}}\left(\vec{r}\cdot\dot{\vec{r}}\right)\right] - \mu\frac{\vec{r}}{r} \tag{32}$$

or

The Eccentricity Vector

$$\boxed{\vec{e} = \frac{1}{\mu}\left[\left(V^2 - \frac{\mu}{r}\right)\vec{r} - \left(\vec{r}\cdot\vec{V}\right)\vec{V}\right] \qquad (33)}$$

<u>Orbit Determination Algorithm</u> (Given $\vec{r}$, $\vec{V}$, and t)

We will assume that we are given or can determine the position and velocity vectors in terms of the classic inertial coordinates:

Given: $\vec{r} = X\hat{i} + Y\hat{j} + Z\hat{k}$ and $\vec{V} = \dot{X}\hat{i} + \dot{Y}\hat{j} + \dot{Z}\hat{k} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

1. Determine the magnitudes of the position and velocity vectors:

$$r = \sqrt{\vec{r}\cdot\vec{r}} \qquad\qquad V = \sqrt{\vec{V}\cdot\vec{V}}$$

2. Determine the ***energy equation*** and the ***semi-major axis***:

$$En = \frac{V^2}{2} - \frac{\mu}{r} \quad \Rightarrow \quad a = -\frac{\mu}{2En} \tag{34}$$

However, since there is a remote possibility of a parabolic orbit (or a near parabolic orbit) one usually calculates the quantity (from the energy equation):

$$\frac{r}{a} = \frac{r V^2}{\mu} - 2 \qquad \text{or} \qquad \frac{1}{a} = -\frac{2\,En}{\mu} \tag{35}$$

Then, if $a = \infty$, either quantity will equal zero and the algorithm will continue.

3. Compute the ***angular momentum vector*** and its magnitude:

$$\vec{h} = \vec{r} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X & Y & Z \\ \dot{X} & \dot{Y} & \dot{Z} \end{vmatrix} = \begin{Bmatrix} Y\dot{Z} - Z\dot{Y} \\ Z\dot{X} - X\dot{Z} \\ X\dot{Y} - Y\dot{X} \end{Bmatrix} = \begin{Bmatrix} h_x \\ h_y \\ h_z \end{Bmatrix} \tag{36}$$

$$h = \sqrt{\vec{h} \cdot \vec{h}} \qquad\qquad p = \frac{h^2}{\mu}$$

4. Compute the ***eccentricity vector*** and its magnitude::

$$\vec{e} = \frac{1}{\mu}\left[\vec{V} \times \vec{h} - \mu \frac{\vec{r}}{r}\right] = \frac{1}{\mu}\left[\left(V^2 - \frac{\mu}{r}\right)\vec{r} - \left(\vec{r} \cdot \vec{V}\right)\vec{V}\right] \tag{37}$$

$$e = \sqrt{\vec{e} \cdot \vec{e}} \qquad \text{or} \qquad e = \sqrt{1 + \frac{2\,h^2\,En}{\mu^2}}$$

The last two equations can be used as a check on progress.

5. Compute the ***inclination:***

$$\cos i = \frac{\vec{h} \cdot \hat{k}}{h} = \frac{h_z}{h} \qquad \begin{array}{ll} \cos i \geq 0 & 0 \leq i \leq 90 \text{ deg} \\ \cos i < 0 & 90 < i \leq 180 \text{ deg} \end{array} \tag{38}$$

The inclination always lies between 0 and 180 deg. For $i > 90$ deg the orbit is ***retrograde***

6. Compute the ***nodal vector*** and its magnitude:

$$\vec{n} = \hat{k} \; x \; \vec{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ h_x & h_y & h_z \end{vmatrix} = \begin{Bmatrix} -h_y \\ h_x \\ 0 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{39}$$

$$n = \sqrt{\vec{n} \cdot \vec{n}}$$

7. Compute the ***longitude of the ascending node*** (***right ascension of the ascending node***)

$$\cos\Omega = \frac{\vec{n} \cdot \hat{i}}{n} = \frac{n_x}{n} \qquad \text{if} \quad \begin{matrix} n_y > 0 & \Omega < 180 \text{ deg} \\ n_y < 0 & \Omega > 180 \text{ deg} \end{matrix} \tag{40}$$

Here, $\cos\Omega$ can be positive or negative. If positive, Ω must lie in the first or fourth quadrant, and if negative, in the second or third. This fact, plus the information gained from the sign of n_y should be enough to determine which is the correct quadrant for Ω.

8. Compute the ***argument of periapsis***:

$$\cos\omega = \frac{\vec{n} \cdot \vec{e}}{n\,e} \qquad \text{if} \quad \begin{matrix} e_z > 0 & \omega < 180 \text{ deg} \\ e_z < 0 & \omega > 180 \text{ deg} \end{matrix} \tag{41}$$

The quadrant for ω can be determined in a similar manner as for the longitude of the ascending node, check e_z to see in which half ω exists, then the sign of $\cos\omega$ to determine the quadrant.

9. Determine the ***true anomaly***:

$$\cos\nu = \frac{\vec{r} \cdot \vec{e}}{r\,e} \qquad \text{if} \quad \begin{matrix} \vec{r} \cdot \vec{V} > 0 & \nu < 180 \text{ deg} \\ \vec{r} \cdot \vec{V} < 0 & \nu > 180 \text{ deg} \end{matrix} \tag{42}$$

Again, the quadrant of the true anomaly can be determined by checking $\vec{r} \cdot \vec{V}$ and determining the half, then checking the sign of $\cos\nu$ to obtain the quadrant.

10. Determine the time of periapsis passage:

Often times we can stop at step 9 since we know just about everything we need to about the orbit. However, if we want to determine the sixth element, the time of periapsis passage we need to use the current time, and Kepler's equation. We can determine the eccentric anomaly

from:

$$r = a(1 - e\cos E) \qquad \text{or} \qquad \tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2} \tag{43}$$

Again one must be careful to get the right quadrant. The rule here is that the true and eccentric anomalies are always in the same half of the orbit. Once we determine the eccentric anomaly, we can solve for the time of periapsis passage from Kepler's equation:

$$n(t - \tau) = E - e\sin E \qquad n^2 a^3 = \mu \tag{44}$$

If the orbit is circular (and ω is not defined) or if we just want to determine the argument of latitude, we can determine it, in general, as follows:

$$\cos u = \frac{\vec{r}\cdot\vec{n}}{rn} \qquad \text{if} \qquad \begin{array}{ll} \vec{r}\cdot\hat{k} < 0 & u < 180 \text{ deg} \\ \vec{r}\cdot\hat{k} > 0 & u > 180 \text{ deg} \end{array} \tag{45}$$

or if $e \neq 0$, $u = \omega + \nu$.

<u>Special Cases</u>

As indicated previously, if the eccentricity and/or the inclination is zero, then some of the classic orbital elements to do not exist. If this is the case, then it will show up in our algorithm by finding zeros in the denominator of some of the terms. We an summarize some of these statements in a table:

Quantity	Doesn't Exist If:
$\cos\Omega = \dfrac{\hat{i}\cdot\vec{n}}{n}$	n = 0
$\cos\omega = \dfrac{\vec{n}\cdot\vec{e}}{ne}$	n = 0 e = 0
$\cos\nu = \dfrac{\vec{r}\cdot\vec{e}}{re}$	e = 0

Case $n = 0$, $e \neq 0$:
Longitude of periapsis:

$$\cos\varpi = \frac{\hat{i}\cdot\vec{e}}{e} \qquad \text{if} \qquad \begin{array}{ll} \vec{e}\cdot\hat{j} > 0 & \varpi < 180 \text{ deg} \\ \vec{e}\cdot\hat{j} < 0 & \varpi > 180 \text{ deg} \end{array} \tag{46}$$

otherwise if $n \neq 0$ and $e \neq 0$ $\qquad \varpi = \Omega + \omega$

Case $n \neq 0$, $e = 0$
Argument of latitude:

$$\cos u = \frac{\vec{r} \cdot \vec{n}}{r\,n} \qquad \text{if} \qquad \begin{array}{ll} \vec{r} \cdot \hat{k} < 0 & u < 180 \text{ deg} \\ \vec{r} \cdot \hat{k} > 0 & u > 180 \text{ deg} \end{array} \tag{47}$$

otherwise if $n \neq 0$ and $e \neq 0$ $\qquad u = \omega + \nu$

Case $n = 0$, $e = 0$
True longitude:

$$\cos \ell = \frac{\hat{i} \cdot \vec{r}}{r} \qquad \text{if} \qquad \begin{array}{ll} \vec{r} \cdot \hat{k} < 0 & \ell > 180 \text{ deg} \\ \vec{r} \cdot \hat{k} > 0 & \ell < 180 \text{ deg} \end{array} \tag{48}$$

Otherwise if $n \neq 0$ and $e \neq 0$, $\qquad \ell = \Omega + \omega + \nu = \varpi + \nu$

Example:

Given: $\vec{r} = -\sqrt{1/2}\,\hat{i} + \sqrt{1/2}\,\hat{j}$ DU; $\qquad \vec{V} = 1/2\,\hat{j}$ DU/TU

1. Determine magnitudes:

$$r = \sqrt{\vec{r} \cdot \vec{r}} = \left(\left(\sqrt{1/2}\right)^2 + \left(\sqrt{1/2}\right)^2\right)^{1/2} = 1 \text{ DU}; \qquad V = 1/2 \text{ DU/TU}$$

2. Calculate Energy and Semi-major axis:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{1} = -\frac{7}{8} = -0.875\ \frac{\text{DU}^2}{\text{TU}^2}$$

$$\frac{1}{a} = -\frac{2\,En}{\mu} = -\frac{2\,(-7/8)}{1} = \frac{7}{4} = 1.75 > 0 \quad \text{ellipse} \qquad a = 4/7 = 0.5714 \text{ DU}$$

3. Calculate angular momentum and its magnitude:

$$\vec{h} = \vec{r} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sqrt{1/2} & \sqrt{1/2} & 0 \\ 0 & 1/2 & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1/2\sqrt{1/2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.3535 \end{Bmatrix}$$

$$h = -1/2\sqrt{1/2} = -0.3535 \ \mathrm{DU}^2/\mathrm{TU}$$

4. Compute the eccentricity vector and its magnitude:

$$\vec{e} = \frac{1}{\mu}\left[\left(V^2 - \frac{\mu}{r}\right)\vec{r} - \left(\vec{r}\cdot\vec{V}\right)\vec{V}\right] = \frac{1}{1}\left[\left(\frac{1}{4} - \frac{1}{1}\right)\begin{Bmatrix} -\sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{Bmatrix} - \left(1/2\sqrt{1/2}\right)\begin{Bmatrix} 0 \\ 1/2 \\ 0 \end{Bmatrix}\right]$$

$$\vec{e} = \begin{Bmatrix} 3/4\sqrt{1/2} \\ -\sqrt{1/2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.5303 \\ -0.7071 \\ 0 \end{Bmatrix} \qquad e = \left[1/2\,(3/4)^2 + 1/2\right]^{1/2} = 0.8839$$

or

$$e = \sqrt{1 + \frac{2\,h^2\,En}{\mu^2}} = \sqrt{1 + \frac{2\,(0.125)\,(-0.875)}{1^2}} = 0.8839$$

5. Compute the inclination:

$$\cos i = \frac{\vec{h}\cdot\hat{k}}{h} = h + \frac{z}{h} = \frac{-0.3535}{0.3535} = -1 \qquad i = 180 \ \mathrm{deg} \quad \text{(Retrograde)}$$

6. Compute the nodal vector:

$$\vec{n} = \hat{k} \times \vec{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ h_x & h_y & h_z \end{vmatrix} = \begin{Bmatrix} -h_y \\ h_x \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad n = |\vec{n}| = 0$$

Since n = 0, then the longitude of the ascending node and argument of periapsis are undefined:

7. Calculate longitude of the ascending node:

$$\cos\Omega = \frac{n_x}{n} = \frac{0}{0} \qquad \text{undefined}$$

8. Calculate argument of periapsis:

$$\cos\omega - \frac{\vec{n}\cdot\vec{e}}{n\,e} = \frac{0}{0} \qquad \text{undefined}$$

Since n = 0 we can calculate the longitude of the periapsis

$$\cos\varpi = \frac{\hat{i}\cdot\vec{e}}{e} = \frac{e_x}{e} = \frac{0.5303}{0.8839} = 0.6000 \quad \Rightarrow \quad \varpi = \pm 153.13 \text{ deg } (153.13 \text{ or } 306.87 \text{ deg})$$

However, $\vec{e}\cdot\hat{j} = e_y < 0$ so $\varpi > 180$ deg. Since the cosine of $\varpi > 0$, the angle must be in the fourth quadrant: <u>ϖ = 306.87 degrees</u>

9. Calculate the true anomaly:

$$\cos\nu = \frac{\vec{r}\cdot\vec{e}}{r\,e} = \frac{\left(-\sqrt{1/2}\,\hat{i} + \sqrt{1/2}\,\hat{j}\right)\cdot\left(0.5303\,\hat{i} - 0.7071\,\hat{j}\right)}{(1)\,(0.8939)} = \frac{-0.8750}{0.8839} = -0.9899$$

$\nu = \pm\ 171.85\ ^\circ\ (171.85 \text{ or } 188.15 \text{ deg})$ Since $\vec{r}\cdot\vec{V} > 0$, ν must be in first half, hence:

<u>ν = 171.85 deg</u>