Summary of Patched Conic Approximations

The patched conic approximation for interplanetary transfers assumes that the sphere of influence of a planet has an infinite radius when observed from the planet, and has zero radius when observed from the Sun. Trajectories within the sphere of influence are two body problems with the planet as the primary attracting body, and trajectories outside the sphere of influence are two body problems with the Sun as the primary attracting body. The conditions going from planetocentric orbit to the heliocentric orbit or vice-versa are called the "patch conditions." The key patch condition is to relate the velocity of the vehicle with respect to the planet to the velocity of the vehicle with respect to the Sun. The difference of the two is the velocity of the planet with respect to the Sun. Since velocities are vectors, we need to account for magnitudes and directions.

In order to discuss the problem in detail, we will define the following locations which may be considered the patch points:

1. Conditions leaving the first planet
2. Conditions arriving at the target planet
3. Conditions on leaving the target planet (fly-by)

In addition we will make the following (simplifying) assumption: **The planet orbits are circular**

The following equations deal with the various orbits and the patch points. The initial calculation is always that regarding the heliocentric transfer orbit. In particular we need to calculate the initial radius and velocity of the transfer orbit (point 1) and the final radius and velocity of the transfer orbit (point 2). Further, we need to calculate the $\Delta V$ required to takes us from the initial planet's circular heliocentric orbit into the transfer orbit, $\Delta V_1$, and the $\Delta V$ required to take us out of the heliocentric transfer orbit and into the target planet's circular heliocentric orbit, $\Delta V_2$.

**<u>Heliocentric Orbit Calculations:</u>**

Define: $\vec{V}_c$ The circular velocity of the planet with respect to the Sun

$\vec{V}$ The heliocentric orbit velocity

$\Delta\vec{V}$ The difference of the planet circular velocity and the heliocentric orbit velocity.

$\phi$ The flight path angle of $\vec{V}$ (Since the planet orbits are assumed circular, this angle is the angle between $\vec{V}_c$ and $\vec{V}$)

Also designate the magnitude of these quantities as $|\vec{V}_c| = V_c$, $|\vec{V}| = V$, and $|\Delta\vec{V}| = \Delta V$. Then we can write the following equations for the departure from the planet 1 heliocentric orbit into the transfer orbit, and from the transfer orbit into the planet 2 heliocentric orbit.

At planet 1:

$$\Delta V_1^2 = V_{c_1}^2 + V_1^2 - 2\, V_{c_1} V_1 \cos \phi_1 \tag{1}$$

At planet 2:

$$\Delta V_2^2 = V_{c_2}^2 + V_2^2 - 2\, V_{c_2} V_2 \cos \phi_2 \tag{2}$$

The values in these equations come from the Heliocentric transfer orbit that you select, e.g. Hohmann, or two year orbit, or some other selected orbit.

**<u>Patch Conditions</u>**

We now would like to calculate the properties of the planet-centered orbit. However, we need to find out some properties of this orbit to do so. We get these properties by matching orbits at the "sphere of influence" boundary which is infinite with respect to planet 1, but is zero with respect to the Sun. At the "patch" point we have:

$$\Delta \vec{V} = \vec{V}_\infty \qquad \text{and} \qquad |\Delta \vec{V}| = |\vec{V}_\infty| \qquad \text{or} \qquad \Delta V = V_\infty \tag{3}$$

We now need to change the reference frame for the velocities. These relations are also called the "patch" conditions. We will designate the velocity of the vehicle with respect to the Sun as $\vec{V}_{v/s}$ and use similar notation for the velocity of the vehicle with respect to the planet, $\vec{V}_{v/p}$ and the velocity of the planet with respect to the Sun, $\vec{V}_{p/s}$. We can then write the following patch condition:

$$\vec{V}_{v/s} = \vec{V}_{v/p1} + \vec{V}_{p1/s}$$

$$\textit{or}$$

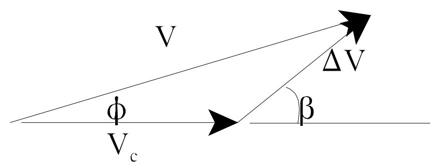
$$\vec{V}_1 = \vec{V}_\infty + V_{c_1}$$

$$\textit{or}$$

$$\vec{V}_1 = \Delta \vec{V}_1 + V_{c_1}$$

with similar equations for planet 2.

The only thing we have to be careful with here is that we use the same units for the velocity terms. Here we will try and be consistent and use $\Delta V$ when dealing with AU/TU, and to use $V_\infty$ when dealing with DU/TU. We now need to carry out these vector operations. It is convenient to use scalar equations and compute magnitudes and directions. We will reference all directions to the velocity vector of the planet with respect to the Sun (the local Sun horizontal in this case since the orbits are assumed circular). The basic figure is shown below and is the same figure that is used to get Eqs. (1) and (2).




The magnitude of $V$ is determined from Eqs. (1) and (2) presented previously. Of interest here is the angle that $\Delta V$ (or $V_\infty$) makes with respect to the planet velocity vector. This is established from:

For Planet 1:

$$Tan\,\beta_1 = \frac{V_1 \sin\phi_1}{V_1 \cos\phi_1 - V_{c_1}} \tag{4}$$

and for Planet 2:

$$Tan\,\beta_2 = \frac{V_2 \sin\phi_2}{V_2 \cos\phi_2 - V_{c_2}} \tag{5}$$

Note that in general, we must look at the sign of the numerator and denominator to determine the quadrant in which $\beta$ is located (+/+) is first, (+/-) is second, (-/-) is third, and (-/+) is fourth. However, if $\phi$ is zero, the angle will be either 0 or 180 degrees. By observing the value of the denominator we can determine the proper angle. If the denominator is (+), the angle is 0, and if the denominator is (-) the angle is 180 degrees.

**Departure Planet (planet 1) Orbit Properties**

The properties of the planet escape orbit can be determined from the information that we have obtained from the heliocentric orbit and the patch conditions, Eq. (3). In addition it is assumed we know the orbit radius at thrust burn out, $r_{bo}$. From the patch condition we have:

$$V_{\infty}\ (\mathrm{DU/TU}) = \frac{\text{heliocentric reference speed}}{\text{planetocentric reference speed}}\ x\ \Delta V\ (\mathrm{AU/TU}) \qquad (6)$$

The heliocentric reference speed is the Earth Mean Orbital Speed (EMOS) and is given by (JGM2) 29.784852 km/s. The planetocentric reference speed depends on the planet gravitational parameter, and the planet mean equatorial radius. It is the orbit speed of a satellite orbiting at the mean equatorial radius:

$$V_{ref} = \sqrt{\frac{\mu}{R}}$$

where $\mu$ is the gravitational parameter of the planet, and $R$ is the mean equatorial radius of the planet

The velocity at infinity with respect to the planet 1 is sufficient to establish the energy of the escape orbit and hence the velocity at any other point in the orbit once the radius is known. From the energy equation:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{V_{\infty}^2}{2} \qquad \Rightarrow \qquad V_{bo}^2 = V_{\infty}^2 + \frac{2\mu}{r_{bo}} \qquad (7)$$

where the energy equation has been evaluated at the burnout radius.

***Insertion into the escape orbit***

It is assumed that the vehicle is in some sort of parking orbit. In most cases it would be in a circular parking orbit at some radius $r_{pk}$. For a legitimate escape with a single burn, it is clear that the periapsis of the escape orbit must less than or equal to the radius of the parking orbit. Or $r_{p_{escape}} \leq r_{pk}$. If they are not equal, they will intersect at some angle that we will call $\phi_{bo}$. Note that the radius of the parking orbit is the burnout radius, $r_{pk} = r_{bo}$Then the $\Delta V$required to leave the parking orbit and enter into the escape orbit is given by the (by now familiar equation:

$$\Delta V_{bo}^2 = V_{bo}^2 + V_{c_{pk}}^2 - 2\, V_{bo}\, V_{c_{pk}} \cos\phi_{bo} \qquad (8)$$

where $V_{c_{pk}} = \sqrt{\dfrac{\mu}{r_{pk}}} = \sqrt{\dfrac{\mu}{r_{bo}}}$, and $V_{bo}$ is determined from Eq. (7). Note that $\phi_{bo}$ is our choice, and enters into the decision where in the parking orbit we apply the $\Delta V_{bo}$ and, as indicated in Eq. (8), the magnitude of $\Delta V_{bo}$. Note that $V_{bo}$ as established from Eq.(7) is fixed and cannot change. The most common (special) case would be where the thrust is applied parallel to the parking orbit and $\phi_{bo} = 0$, and Eq. (8) reduces to $\Delta V_{bo} = V_{bo} - V_{c_{pk}}$. The following equations establish the properties of the escape orbit. They will be written for the most general case, but reduce to give the correct result for the special case of a tangent burn.

From the energy equation, Eq. (7), with $\mathscr{E} = \dfrac{\mu}{2\,|a|}$ we have:

$$|a| = \frac{\mu}{V_\infty^2} \tag{9}$$

The eccentricity of the escape orbit is given by:

$$\begin{aligned} e^2 &= \left( \frac{r_{bo} V_\infty^2}{\mu} + 1 \right)^2 \cos^2 \phi_{bo} + \sin^2 \phi_{bo} \\ &= \left( \frac{r_{bo} V_{bo}^2}{\mu} - 1 \right)^2 \cos^2 \phi_{bo} + \sin^2 \phi_{bo} \end{aligned} \tag{10}$$

Alternatively, if we know the periapsis radius of the escape orbit (even if it is different from the burnout radius) we can get the orbit eccentricity from:

$$e = \frac{r_p}{|a|} + 1 = \frac{r_p V_\infty^2}{\mu} + 1 \tag{11}$$

The true anomaly at burnout is given by:

$$\tan \nu_{bo} = \frac{\dfrac{r_{bo} V_{bo}^2}{\mu} \sin \phi_{bo} \cos \phi_{bo}}{\dfrac{r_{bo} V_{bo}^2}{\mu} \cos^2 \phi_{b0} - 1} = \frac{\left( \dfrac{r_{bo} V_\infty^2}{\mu} + 2 \right) \sin \phi_{bo} \cos \phi_{bo}}{\left( \dfrac{r_{bo} V_\infty^2}{\mu} + 2 \right) \cos^2 \phi_{bo} - 1} \tag{11}$$

The true anomaly at escape, $\nu_\infty$ is given by:

$$\cos \nu_\infty = \frac{1}{e} \tag{12}$$

***Launch Position***

The final calculation we can make is where in the parking orbit we should apply the $\Delta V_{bo}$ to arrive at the proper exit condition. We will measure the launch angle from the velocity vector of the planet, clockwise back to the insertion point. This direction is opposite of all the other angles that are measured clockwise. Here we will use the magnitudes of the previously calculated angles. The launch angle is given by:

$$\theta_L = \nu_\infty + \beta_1 - \nu_{bo} \tag{13}$$

## **Arrival Planet (planet 2) Orbit Properties**

We have calculated the arrival velocity from Eq. (2). We need to change the units to the planet reference units using Eq. (6) to get $V_\infty$ in proper units. If we know the periapsis radius of the entry orbit, we can determine all the properties of the orbit. We generally ***assume that the periapsis radius is given***. This is a parameter that we can use to characterize the entry or flyby orbit. The orbit properties are then calculated from:

$$\mathscr{E} = \frac{V_\infty^2}{2} = \frac{\mu}{2\,|a|} \qquad \Rightarrow \qquad a = \frac{\mu}{V_\infty^2} \tag{14}$$

The eccentricity can be determined from:

$$e = 1 + \frac{r_p V_\infty^2}{\mu} \tag{15}$$

The true anomaly at infinity from:

$$\cos \nu_\infty = -\frac{1}{e} \tag{16}$$

The velocity vector relative to the planet is $\vec{V}_\infty$ and its perpendicular distance to the center of the planet is given by $d$. However the planet attracts the vehicle closer so that the closest approach is the periapsis distance that we mentioned previously. The relation between $d$ and $r_p$ is given by (from angular momentum and energy)

$$d = r_p \sqrt{1 + \frac{2\mu_{planet}}{r_p V_\infty^2}} \tag{17}$$

To achieve the proper distance d, then we must target the heliocentric transfer orbit to intersect the planet's heliocentric orbit a distance $x$ ahead of or behind the target planet, where $x$ is given by:

$$x = \frac{d}{\sin \beta_2} \tag{18}$$

Where x is generally behind the planet if we want an "over-flight" to add energy and is ahead of the planet if we want an "under-flight" to lose energy. Unfortunately Eq. (18) breaks down if $\beta_2 = 0 \ or \ \pi$. Before we resolve this special case, let is define more precisely what we mean by and over or under flight. Looking form the top (north pole) the direction of the orbits about the Sun is counter-clockwise. If the hyperbolic approach (or flyby) orbit goes around the planet in the counter-clockwise direction, we say it is an over-flight. If it goes around the planet in a clockwise direction, we say it is an under-flight. We can now resolve the case where $\beta_2 = 0 \ or \ \pi$. If $\beta_2 = 0 \ or \ \pi$, then the heliocentric orbit must pass either above the planet of below the planet by the amount $d$. If $\beta = 0$, then an over-flight will require the heliocentric orbit to pass above the planet by an amount $d$, and an under-flight will require the heliocentric orbit to pass under the planet by the amount $d$. On the other hand, if $\beta = \pi$, then for an over-flight, the heliocentric orbit must pass below the planet by the amount $d$, and if an under-flight, the heliocentric orbit must pass above the planet by the amount $d$!

***Capture Requirements***

If we desire to capture the vehicle on the target planet 2, then the capture radius must be greater than or equal to the periapsis radius, that is $r_{cap} \geq r_p$. In the entry orbit we can calculate the velocity at any radius from the energy equation

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{V_\infty^2}{2} \qquad \Rightarrow \qquad V_{cap}^2 = V_\infty^2 + \frac{2\,\mu}{r_{cap}} \tag{19}$$

The flight path angle at capture is:

$$\cos\phi = \frac{h}{r_{cap}\,V_{cap}} = \frac{V_\infty\, d}{r_{cap}\,V_{cap}} \tag{20}$$

Next, the true anomaly at capture is given by:

$$\tan\nu_{cap} = \frac{\dfrac{r_{cap}\,V_{cap}^2}{\mu}\sin\phi_{cap}\cos\phi_{cap}}{\dfrac{r_{cap}\,V_{cap}^2}{\mu}\cos^2\phi_{cap} - 1} = \frac{\left(\dfrac{r_{cap}\,V_\infty^2}{\mu} + 2\right)\sin\phi_{cap}\cos\phi_{cap}}{\left(\dfrac{r_{cap}\,V_\infty^2}{\mu} + 2\right)\cos^2\phi_{cap} - 1} \tag{21}$$

The capture burn is given by:

$$\Delta V_{cap}^2 = V_{cap}^2 + V_{c_{pk}}^2 - 2\,V_{cap}\,V_{c_{pk}}\cos\phi_{cap} \tag{22}$$

All these equations reduce for the special case where the capture burn occurs at periapsis, and the final parking orbit has the same radius as the periapsis radius.

Finally we can locate the capture location by describing an angle measured from the planet heliocentric velocity vector similar to how we measured the launch angle previously. The result is:

$$\theta_{cap} = \nu_{inv} + \beta_2 - \nu_{cap} \qquad (23)$$

***Flyby Calculations***

If we decide to fly by the planet, then we will leave the planet with the same velocity as we approached it as observed from the planet (energy is conserved in the flyby orbit). Consequently we arrive back at the sphere of influence with the velocity $\vec{V}_\infty$. It will be oriented in a different direction then on approach. The difference is caused by the turning angle of the hyperbolic orbit. The turning angle is given by:

$$\sin\frac{\delta}{2} = \frac{1}{e} \qquad (24)$$

The angle that $\vec{V}_\infty$ makes with the planet heliocentric velocity vector is designated by $\beta_3$ and is determined from:

$$\beta_3 = \beta_2 \pm \delta \qquad (25)$$

where: the + sign is used for an under-flight (as defined previously)
the - sign is used for an over-flight (as defined previously)

**Patch Conditions on Leaving Planet 2**

The patch conditions on leaving planet 2 will be designated by conditions at point 3. We have:

$$\begin{aligned} \vec{V}_{v/s} &= \vec{V}_{v/p} + \vec{V}_{p/s} \\ \vec{V}_3 &= \vec{V}_\infty + \vec{V}_{c_3} \\ &= \Delta\vec{V}_3 + \vec{V}_{c_2} \qquad\qquad \vec{V}_{c_2} = \vec{V}_{c_3} \end{aligned} \qquad (26)$$

Here we know $\Delta\vec{V}$ and $\vec{V}_c$ and we need to determine $\vec{V}$. Again we will find the magnitude and the angle associated with $\vec{V}_3$. The resulting equations are:

$$V_3^2 = \Delta V_3^2 + V_{c_3}^2 + 2\,\Delta V_3\, V_{c_3} \cos\beta_3 \qquad (27)$$

Here, $\Delta V_3$ was used instead of $V_\infty$ to remind us to be sure to switch units back to AU/TU in this equation. The heliocentric flight path angle for $\vec{V}_3$ is given by:

$$\tan\phi_3 = \frac{\Delta V_3 \sin\beta_3}{\Delta V_3 \cos\beta_3 + V_{c_3}} \qquad (28)$$

**Properties of the Heliocentric Orbit After Flyby**

Since we know the velocity, flight path angle, and radius at point 3, we can determine everything we need to know about the post flyby heliocentric orbit.

***Energy:***

$$\frac{V_3^2}{2} - \frac{\mu}{r_3} = \mathscr{E} = -\frac{\mu}{2a} \qquad (29)$$

***Angular Momentum:***

$$h = r_3\, V_3 \cos\phi_3 \qquad (30)$$

***True anomaly:***

$$\tan\nu_3 = \frac{\dfrac{r_3\, V_3^2}{\mu} \sin\phi_3 \cos\phi_3}{\dfrac{r_3\, V_3^2}{\mu} \cos^2\phi_3 - 1} \qquad (31)$$