Local to Inertial Coordinate Transformation

The transformation from the local coordinate system (x^l aligned with the radius vector, z^l aligned with the angular momentum vector, and y^l in the orbit plane and completing the right hand set) to the inertial coordinate system (X^I aligned with the Vernal Equinox, Z^I aligned with the North pole, and Y^I in the plane of the Equator (or ecliptic) and completing the right hand set) is given by the following matrix:

$$\left\{\vec{A}\right\}^I = \left[T_{l2I}\right]\left\{\vec{A}\right\}^l \qquad \textbf{(1)}$$

where $\{\ \}^l$ means represented in the l (local) system, and $\{\ \}^I$ means represented in the inertial system, $\vec{A}$ is a generic vector, and

$$T_{l2I} = \begin{bmatrix} \cos\Omega\cos(\omega+\nu)-\sin\Omega\cos i\sin(\omega+\nu) & -\cos\Omega\sin(\omega+\nu)-\sin\Omega\cos i\cos(\omega+\nu) & \sin\Omega\sin i \\ \sin\Omega\cos(\omega+\nu)+\cos\Omega\cos i\sin(\omega+\nu) & -\sin\Omega\sin(\omega+\nu)+\cos\Omega\cos i\cos(\omega+\nu) & -\cos\Omega\sin i \\ \sin i\sin(\omega+\nu) & \sin i\cos(\omega+\nu) & \cos i \end{bmatrix} \qquad \textbf{(2)}$$

The transformation from the perifocal coordinate system to the inertial system is given by the above transformation matrix with $\nu = 0$,

$$T_{p2I} = \begin{bmatrix} \cos\Omega\cos\omega-\sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega-\sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega+\cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega+\cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix} \qquad \textbf{(3)}$$

We need not to determine the position and velocity vector in the local coordinate system. Recall that the local x axis is aligned with the radius vector. Therefore any components in the radial direction are local x components and any in the transverse direction are y components. It follows that the position and velocity vectors have the following representation in the local axis system:

$$\vec{r}^{\,l} = \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}, \quad \text{and} \quad \vec{V}^{\,l} = \begin{Bmatrix} V_r \\ V_\theta \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dot{r} \\ r\,\dot{\nu} \\ 0 \end{Bmatrix} \qquad \textbf{(4)}$$

where:

$$\dot{r} = \frac{\mu}{h} e \sin \nu, \qquad \text{and} \qquad r\,\dot{\nu} = \frac{\mu}{h}(1 + e\cos \nu) \tag{5}$$

Components of the position and velocity vectors in the perifocal coordinate system can be determined by doing a simple transformation from the local axes system to the perifocal axes system. This transformation can be done using a simple -ν rotation about the z axis. The transformation is then given by:

$$\{\vec{A}\}^p = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \{\vec{A}\}^l \tag{6}$$

For the position vector we have:

$$\{\vec{r}\}^p = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix}^l = \begin{Bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{Bmatrix}^p \tag{7}$$

And for velocity we have:

$$\{\vec{V}\}^p = \begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\mu}{h} e\sin\nu \\ \frac{\mu}{h}(1 + e\cos\nu) \\ 0 \end{Bmatrix}^l = \begin{Bmatrix} -\frac{\mu}{h}\sin\nu \\ \frac{\mu}{h}(e + \cos\nu) \\ 0 \end{Bmatrix}^p \tag{8}$$

Equations (7) and (8) give the position and velocity in the peri-focal coordinate system.