Orbit Characteristics

We have shown that the in the two body problem, the orbit of the satellite about the primary (or vice-versa) is a conic section, with the primary located at the focus of the conic section. Hence the orbit is either an ellipse, parabola, or a hyperbola, depending on the eccentricity. For conic sections we have the following classifications:

$e < 1$	ellipse	
$e = 1$	parabola	(An exception this relation is that all rectilinear orbits have e = 1, and angular momentum = 0)
$e > 1$	hyperbola	

Of main interest for Earth centered satellites (Geocentric satellites) and Sun centered satellites (Heliocentric satellites) are elliptic orbits. However when we go from one regime to another such as leaving the Earth and entering into an interplanetary orbit then we mus deal with hyperbolic orbits. Or if we approach a planet form a heliocentric orbit, hyperbolic orbits are of interest. Parabolic orbits, on the other hand are more theoretical than practical and simply define the boundary between those orbits which are periodic and "hang around," (elliptic orbits), and those orbits which allow one to escape from the system (hyperbolic orbits). So one might say that the parabolic orbit is the minimum energy orbit that allows escape. In the following, we will determine the properties of each of these types of orbits and write some equations that are applicable to all orbits and other equations that are applicable only to the type of orbit of interest.

It was indicated earlier that the constant e (eccentricity) was related to some other constants (angular momentum and energy that we developed earlier). We can now determine this relation by using the expression we found for r and substitute it into the energy equation. The energy equation is given by

$$\frac{V^2}{2} - \frac{\mu}{r} = En$$
$$\frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \frac{\mu}{r} = En \qquad \textbf{(1)}$$

However, since the energy is a constant over the whole orbit, we can evaluate the energy equation a any convenient point we choose. If you pick an arbitrary point in the orbit, the result will be the same. To reduce the algebra required, we can select the periapse location and evaluate Eq. 1 at periapse. At periapse, the radial component of velocity is zero, hence we eliminate a term. Also a periapse, we can note that r becomes,

$$r = r_{\min} = \left.\frac{h^2/\mu}{1 + e\cos\nu}\right|_{\nu=0} = \frac{h^2/\mu}{1+e} \qquad \textbf{(2)}$$

From the angular momentum relation $r^2\dot{\theta} = h$, we can replace $\dot{\theta}$ with h/r^2. Equation 1 becomes,

$$\frac{h^2}{r^4} - \frac{\mu}{r^2} = En \qquad \textbf{(3)}$$

We can replace r with Eq. 2 (r at periapse) to get

$$\frac{h^2 (1 + e)^2}{2 (h^2/\mu)^2} - \frac{\mu (1 - e)}{h^2/\mu} = En \qquad \textbf{(4)}$$

If we simplify we end up with the desired result,

$$e = \sqrt{1 + \frac{2 h^2 En}{\mu^2}} \qquad \textbf{(5)}$$

Equation 5 relates the eccentricity of the orbit to its angular momentum and energy. It is valid for any type of orbit. It is clear from Eq. 5 that the following classification can occur.

En < 0	e < 1	orbit is ellipse
En = 0	e = 1	orbit is parabola
En > 0	e > 1	orbit is a hyperbola

These are important relations that should be committed to memory. Eq. 5 is also an important equation to know.

Summary of General Results for All Orbits (elliptic, parabolic, and hyperbolic)

The Equation of the orbit:

$$r(\nu) = \frac{h^2/\mu}{1 + e\cos\nu} = \frac{p}{1 + e\cos\nu} \qquad \textbf{(6)}$$

where $p = h^2/\mu$ = Orbit Parameter
r = radial distance from center of attracting body (focus)
e = eccentricity
ν = true anomaly (measured from point of closest approach (apoapse)

Energy Equation:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{V_r^2 + V_\theta^2}{2} - \frac{\mu}{r} = En \qquad \textbf{(7)}$$

where: $V_r = \dot{r}$, $V_\theta = r\,\dot{\nu}$

Angular Momentum:

$$h = r^2\,\dot{\nu} = r\,V_\theta = r\,V\cos\phi = const \qquad \textbf{(8)}$$

where ϕ (or γ) is the flight path angle (angle velocity vector makes with local horizontal).

We can develop other relations as follows.

Radial Velocity in Terms of True Anomaly

The radial velocity is just $\dot{r}$, which can be obtained by differentiating Eq. 6 with respect to time.

$$\dot{r} = \frac{p\,e\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)^2} = \frac{r\,\dot{\nu}\,e\sin\nu}{1 + e\cos\nu} \qquad \textbf{(9)}$$

By multiplying and dividing by p = h^2/μ we can rewrite Eq. 9 as,

$$\dot{r} = \frac{1}{p}\,r^2\,\dot{\nu}\,e\sin\nu = \frac{\mu}{p}\,e\sin\nu \qquad \textbf{(10)}$$

From the basic definition of the flight path angle and from Eq. 9 we have,

$$Tan\,\phi = \frac{V_r}{V_\theta} = \frac{\dot{r}}{r\,\dot{\nu}} = \frac{e\sin\nu}{1 + e\cos\nu} \qquad \textbf{(11)}$$

We can also define the ***semi-latus rectum*** as the value of r when $\nu = \pi/2$. Hence the semi-latus rectum = p = $h^2/2$. Finally we can note that since cos ν = cos -ν, that the orbit is symmetric about the point of closest approach (or about the major axis of the conic section).

PROPERTIES OF THE ORBITS

<u>Parabolic Orbit (e = 1, En = 0)</u>

The parabolic orbit serves as a boundary between the elliptic (periodic) orbits and the hyperbolic (escape) orbits. It is the orbit of least energy that allows escape. The orbit equation becomes,

$$r(\nu) = \frac{h^2/\mu}{1 + \cos\nu} = \frac{p}{1 + \cos\nu} \qquad \textbf{(12)}$$

Further the periapse distance = r(0) = p/2 = h^2 / 2μ.

The energy equation becomes.

$$\frac{V^2}{2} - \frac{\mu}{r} = 0 \quad \Rightarrow \quad V_{escape} = \sqrt{\frac{2\mu}{r}} \qquad \textbf{(13)}$$

Equation 13 defines the ***escape velocity***, which is the minimum velocity to escape the two body system at the given radius r. Note that the speed required for escape is independent of it direction!

The flight path angle in a parabolic orbit is given by,

$$\tan\phi = \frac{\sin\nu}{1 + \cos\nu} = \frac{2\sin\nu/2\,\cos\nu/2}{1 + \cos^2\nu/2 - \sin^2\nu/2} = \tan\nu/2$$

$$\text{or} \qquad \textbf{(14)}$$

$$\phi = \frac{\nu}{2}$$

Also it is easy to show (use V cos ϕ = r $\dot{\nu}$) that $\cos\phi = \sqrt{\dfrac{h^2}{2\mu r}} = \sqrt{\dfrac{r_p}{r}}$.

<u>Elliptic Orbits (e < 1, En , 0)</u>

Recall that the radius position is measured from the focus of the ellipse (not the center). By evaluating the orbit equation at values of the true anomaly of 0 and π, we can determine the closest approach (periapse distance, r_p) and the furthest approach (apoapse distance, r_a). The sum of these two comprise the ***major axis*** of the ellipse. Of interest to us is half the distance or the ***semi-major axis***, usually designated by the symbol ***a***. Hence *a* is the distance from the center of the ellipse to the periapse and apoapse. We can set ν = 0 and π to obtain r_p and r_a respectively,

$$r(0) = r_{\min} = r_p = \frac{h^2/\mu}{1+e}, \qquad r(\pi) = r_{\max} = r_a = \frac{h^2/\mu}{1-e} \tag{15}$$

Then the major axis, $2a = r_a + r_p$, and we have,

$$2a = h^2/\mu\left[\frac{1}{1+e} + \frac{1}{1-e}\right] = h^2/\mu\left[\frac{2}{1-e^2}\right] \tag{16}$$

or $$\boxed{h^2/\mu = a(1-e^2)}.$$

Hence the orbit equation for the ellipse becomes;

$$\boxed{r(\nu) = \frac{a(1-e^2)}{1 + e\cos\nu}} \tag{17}$$

Also it follows:

$r_p = a (1 - e)$

$r_a = a (1 + e)$

$a - r_p = a e$

$$e = \frac{r_a - r_p}{r_a + r_p}$$

$b = a\sqrt{1-e^2}$ (semi-minor axis)

<u>Energy in an Elliptic Orbit</u>

We can eliminate the eccentricity from Eq. 16b using Eq. 5,

$$h^2/\mu = a(1-e^2) = a\left[1 - \left(1 + \frac{2h^2 En}{\mu^2}\right)\right] = -\frac{2ah^2 En}{\mu^2} \tag{18}$$

or

$$En = -\frac{\mu}{2a} \tag{19}$$

Hence the energy equation takes the form (for an elliptic orbit)

$$\boxed{\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}} \tag{20}$$

Additionally we have the flight path angle relation

$$\tan\phi = \frac{e\sin\nu}{1 + e\cos\nu} \qquad \textbf{(21)}$$

and in terms of r,

$$\cos\phi = \frac{r\,\dot{\nu}}{V} = \sqrt{\frac{a\,(1 - e^2)}{r\left[2 - \frac{r}{a}\right]}} \qquad \textbf{(22)}$$

An interesting set of relations happen at the point on the orbit at the end of the semi-minor axis. At that point, the following relations can be found:

r = a, cos ν = -e, $\tan\phi = \dfrac{e}{\sqrt{1 - e^2}}$, sin ϕ = e .

<u>Period of an Elliptic Orbit</u>

An elliptic orbit is the only type of orbit that has a period. We can determine the period by recalling the angular momentum equation and noting that half the angular momentum is the aerial rate,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\nu} = \frac{h}{2} \qquad \textbf{(23)}$$

Then integrating both sides over one period or once around the orbit we have

$$\int_0^A dA = \int_0^{T_p} \frac{h}{2} dt = \frac{h}{2} T_p = \pi\,a\,b \quad \text{(area of ellipse)} \qquad \textbf{(24)}$$

But $h = \sqrt{\mu\,a\,(1 - e^2)}$, and $b = a\sqrt{(1 - e^2)}$, and the (1 - e^2) terms cancel leading to the period given as

$$T_p = 2\pi\sqrt{\frac{a^3}{\mu}} \qquad \textbf{(25)}$$

We can see that both the energy and the period of the orbit depend only on the size of the orbit and not on the shape (e).

It is useful to define the ***mean angular rate,*** $n = 2\pi / T_p$. With this definition we can write a form of Kepler's third law, the square of the period of an orbit is proportional to the cube of the orbit size.

$$n^2 a^3 = \mu \qquad \textbf{(26)}$$

Properties of Elliptic Orbits in Term s of Periapse and Apoapse Distances

From geometry we noted previously that $2a = r_a + r_p$. Then the energy equation can be written as

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{r_a + r_p} = En \qquad \textbf{(27)}$$

If we evaluate Eq. 27 at periapse and apoapse, we can determine the velocities at these points. The results, after very little algebra, are,

$$V_p = \sqrt{\frac{\mu}{r_p}} \sqrt{\frac{2\,\frac{r_a}{r_p}}{1 + \frac{r_a}{r_p}}} \qquad V_a = \sqrt{\frac{\mu}{r_a}} \sqrt{\frac{2}{1 + \frac{r_a}{r_p}}} \qquad \textbf{(28)}$$

Since $h = r_p V_p = r_a V_a$ we can calculate h in terms of r_p and r_a, that gives

$$h = \sqrt{\mu\, r_p} \sqrt{\frac{2\,\frac{r_a}{r_p}}{1 + \frac{r_a}{r_p}}} = \sqrt{\mu\, ra} \sqrt{\frac{2}{1 + \frac{r_a}{r_p}}} = \sqrt{\frac{2\,\mu\, r_p\, r_a}{r_p + r_a}} \qquad \textbf{(29)}$$

Circular Orbit - As a Special Case of Elliptic Orbit (e = 0, $En = -\frac{\mu}{2\,r_c}$)

With e = 0, the orbit equation gives,

$$r = p = h^2/\mu = a = const = r_c \qquad \textbf{(30)}$$

From the energy equation,

$$\frac{V_c^2}{2} - \frac{\mu}{r_c} = -\frac{\mu}{2\, r_c} \qquad \Rightarrow \qquad V_c = \sqrt{\frac{\mu}{r_c}} \qquad \textbf{(31)}$$

V_c is called the circular speed and is ***defined*** at every radius r as $V_c = = \sqrt{\frac{\mu}{r}}$ regardless of the orbit.

Hyperbolic Orbits (e > 1, En > 0)

From the orbit equation, it is clear that the radial distance will go to infinity when the denominator term goes to zero.

$r \mapsto \infty$ when $(1 + e \cos \nu) = 0$

Then

$$\nu_{\substack{min \\ max}} = \cos^{-1}\left(-\frac{1}{e} \right) \qquad \textbf{(32)}$$

Equation 32 puts limits on the true anomaly of an hyperbolic orbit.

Just as in an elliptic orbit, we can calculate the closest approach at $\nu = 0$. We can also calculate the "apoapse" distance by letting $\nu = \pi$. However the result is negative and represents the "closest approach" of the other branch of the hyperbola, one that has no meaning in our orbit. However formally we can then note that

$$r_p = \frac{h^2/\mu}{1 - e}, \qquad r_\pi = \frac{h^2/\mu}{1 + e} = -(r_p + 2a) \qquad \textbf{(33)}$$

Substituting for r_p in the above equation yields the hyperbolic orbit result,

$$h^2/\mu = a\,(e^2 - 1) \qquad \textbf{(34)}$$

and the corresponding hyperbolic orbit equation

$$r(\nu) = \frac{a\,(e^2 - 1\,)}{1 + e \cos \nu} \qquad \textbf{(35)}$$

If we substitute for $e^2 = 1 + \frac{2\, h^2\, En}{\mu^2}$ in Eq. 34 we find the hyperbolic orbit energy,

$$En = \frac{\mu}{2a} \quad \textbf{(36)}$$

The corresponding energy equation is

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \quad \textbf{(37)}$$

One of the consequences of Eq. 37 is that at infinity, the velocity is no longer zero and is given by

$$V_\infty = \sqrt{\frac{\mu}{a}} \quad \textbf{(38)}$$

and is defined as the ***hyperbolic excess velocity.***

We can also determine the flight path angle in terms of ν or in terms of r in a similar manner as we did for elliptic orbits. The results are

$$\tan\phi = \frac{e\sin\nu}{1 + e\cos\nu} \quad \textbf{(39)}$$

and

$$\cos\phi = \frac{h}{rV} = \frac{a(e^2 - 1)}{r\left[2 + \frac{r}{a}\right]} \quad \textbf{(40)}$$

` Finally, the hyperbolic orbit has a property that no other orbit has. A vehicle traveling the length of the orbit will arrive coming from some point at infinity, and then fly by through the closest approach point (apoapse), and then leave, going to some point at infinity. The approach direction comes in from a direction of $-\nu_\infty$ and leave in the direction of $+\nu_\infty$. The angle through which the vehicle turns is called the ***turning angle,*** δ. This turning angle can be determined from the properties of the hyperbola.

$$\delta = 2\nu_\infty - \pi \qquad \Rightarrow \qquad \frac{\delta}{2} = \nu_\infty - \frac{\pi}{2} \quad \textbf{(41)}$$

Then $\sin\frac{\delta}{2} = \sin\nu_\infty \cos\frac{\pi}{2} - \cos\nu_\infty \sin\frac{\pi}{2} = -\cos\nu_\infty = \frac{1}{e}$. Consequently the turning angle is given by

$$\sin\frac{\delta}{2} = \frac{1}{e} \qquad \textbf{(42)}$$

The properties characteristic of all orbits are presented in this section. Equations which apply to all orbits are given and those which are applicable to specific orbits are presented.