AOE 5234 Orbital Mechanics Problem Sheet 3

Due 15 February, 2001

1. In class we solved the differential equation of the orbit $\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$, using a Sundman transformation dt = c r^n dθ, with n = 1. The result transformed the above equation into several linear ordinary differential equations which were easily solved. Associated with these equations was the independent variable "anomaly," χ, S, or E, depending upon the value of the constant c. If we consider the independent variable to be the true anomaly, ν, it is easy to show that the angular momentum being constant leads to a Sundman transformation with c = 1/h and n = 2:

$$h = r^2\dot{\theta} \quad \Rightarrow \quad dt = \frac{r^2}{h}d\theta$$

For the general case, let the transformation of interest be $dt = c\,r^2\,d\eta$, so that $\frac{d(\bullet)}{dt} = \frac{1}{cr^2}\frac{d(\bullet)}{d\eta}$. By applying this transformation to the differential equation of the orbit above, derive the following relations:

$$u'' + c^2 h^2 u = c^2 \mu$$

$$\hat{r}'' + c^2 h^2 \hat{r} = 0$$

$$t' = \frac{c}{u^2}$$

where, $u = \frac{1}{r}$, $\quad and \quad \hat{r} = \frac{\vec{r}}{r}$.

Hints: Write the position vector $\vec{r} = r\hat{r} = \frac{1}{u}\hat{r}$. Determine the second equation first. Note that $\vec{r}\cdot\dot{\vec{r}} = r\dot{r}$. $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$. An intermediate step is to show $\frac{d\hat{r}}{d\eta} = c\left(\vec{h}\times\hat{r}\right)$. Applying the transformation to the vector differential equation of the orbit will result in a bunch of terms, all containing the unit vector $\hat{r}$, and, as a result, we can set the coefficients equal to give us the scalar equation in u indicated above.

2. For the case where c = 1/h, write down the three differential equations above. Note that two of them are linear, while the time equation is not.

a) Write the solution to the u equation assuming that when η = 0, u = u_0, and u' = u'_0

b) Write the solution to the $\hat{r}$ equation assuming that when $\eta = 0$, $\hat{r} = \hat{r}_0$ and $\hat{r}' = \hat{r}_0'$.

c) Write a step by step procedure for solving the problem: given $\vec{r}_0$ *and* $\vec{V}_0$ and η,

(the change in true anomaly), find $\vec{r}$ *and* $\vec{V}$.

d) Consider the case where $\vec{r}_0 = [\,0.2, 1.0, 0.3]^T$ and $\vec{V}_0 = [\,1.0, 0.6, 0.2]^T$ and $\eta = 120$ deg

i) Find $\vec{r}$ *and* $\vec{V}$.
ii) Find a, e, h
iii) Find initial true anomaly ν_0
iv) Find initial Eccentric anomaly E_0

Note: In these units, the value of $\mu = 1$.

3. In class we looked at the case where we picked c to be $\sqrt{\dfrac{a}{\mu}}$ and the associated anomaly was E.

Using the solution we had for $\vec{r}$ (E), for the case where the "0" conditions were at periapse

a) Determine the solution for $\vec{V}$ (E) in perifocal coordinates

b) For the general case where $\vec{r}_0$ *and* $\vec{V}_0$ are arbitrary, gather the coefficients of

$\vec{r}_0$ *and* $\vec{V}_0$ and verify that the f and g functions are given by:

$$f = 1 - \frac{a}{r_0}(1 - \cos\Delta E)$$

$$g = t - t_0 - \sqrt{\frac{a^3}{\mu}}(\Delta E - \sin\Delta E)$$

$$\dot{f} = -\frac{\sqrt{\mu a}}{r\,r_0}\sin\Delta E$$

$$\dot{g} = 1 - \frac{a}{r}(1 - \cos\Delta E)$$

Note that you could also take components of the initial and final values of position and velocity and determine the f and g functions as indicated in class and get the same results!

4. Solve the Kepler problem, given initial position and velocity vectors and time of flight, determine the final position and velocity vectors. Use Newton's method (or some math package) for solving the time equation. You can either use the general form ($t - t_0$) and ($E - E_0$) or use the simplified form ($t - \tau$) and calculate ($t_0 - \tau$) and ($t - \tau$) from Kepler's Equation and subtract to get ($t - t_0$). The initial condition and time of flight are:

$\vec{r} = 0.04747\hat{i} + 0.7361\hat{j} + 0.3543\hat{k}$ Distance Units (DU)

and $\vec{V} = -1.1510\hat{i} + 0.0223\hat{j} + 0.3435\hat{k}$ Speed Units (SU), with a time of flight of 3.5 Time Units (TU). Note in this system of units, the gravitational constant is $\mu = 1\ DU^3/TU^2$.

Once you have determined ΔE, you can use the results of problem 3 to get the final answer.