Solution to Differential Equations of Motion

The vector differential equation of motion which describes the relative motion of a satellite with respect to a “primary” is

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \qquad \textbf{(1)}$$

Equation 1 is a second order ordinary vector differential equation. Once we pick a coordinate system, we can write the representation of Eq. 1 is scalar form. For Cartesian coordinates, for example we can write Eq. 1 in scalar form:

$$\begin{aligned} \ddot{x} &= -\frac{\mu}{r^3}x \\ \ddot{y} &= -\frac{\mu}{r^3}y \\ \ddot{z} &= -\frac{\mu}{r^3}z \end{aligned} \qquad \textbf{(2)}$$

where $r^3 = (x^2 + y^2 + z^2)^{3/2}$. In any case, Eqs. 1 & 2 represent a sixth order dynamic system. We can obtain the solution to the system if we can determine 6 constants of integration! We can extract 4 of these constants by applying standard techniques to Eq. 1 in vector form. These standard techniques include looking at the angular motion or moment equation, and applying the work-energy relation. Both of these techniques and be applied to the Newton equation $\vec{F} = m\vec{a}$, which is essentially the form of Eq. 1. Eq. 1 can loosely be considered to be $\vec{a} = \vec{F}/m$.

<u>Angular Motion</u>

The moment of force is defined to be $\vec{r}\ x\ \vec{F}$ and the corresponding moment of momentum or angular momentum is given as $\vec{r}\ x\ m\vec{a}$. In this problem we have divided through by m, hence the results are called specific angular momentum. Here we can take the cross product of the vector r with Eq. 1,

$$\vec{r}\ x\ \ddot{\vec{r}} = \vec{r}\ x\left(-\frac{\mu}{r^3}\right)\vec{r} = 0 \qquad \textbf{(3)}$$

The last equality holds since $\vec{r}\ x\ \vec{r} = 0$. The first term can be rewritten, and Eq. 3 becomes,

$$\frac{d}{dt}\left(\vec{r}\ x\ \dot{\vec{r}}\right) = \frac{d}{dt}\left(\vec{r}\ x\ \vec{V}\right) = 0 \qquad \textbf{(4)}$$

Therefore we can write that the specific angular momentum (angular momentum per unit mass) is a constant which we shall label, $\vec{h}=\text{const}$.

$$\vec{r} \times \vec{V} = \vec{r} \times \dot{\vec{r}} = \vec{h} = \vec{const} \qquad \textbf{(5)}$$

Note that $\vec{h}$ is a vector constant so it is equivalent to 3 scalar constants (3 down, 3 to go).

Specific Energy Constant

We will now extract the work-energy integral. We can extract this integral by noting that $\dot{\vec{r}}\,dt = d\vec{r} = \vec{V}\,dt$. If we take the dot product of this equation with Eq. 1, we have,

$$\ddot{\vec{r}} \cdot \dot{\vec{r}}\,dt = -\frac{\mu}{r^3}\vec{r} \cdot d\vec{r} \qquad \textbf{(6)}$$

Recall, $\vec{A} \cdot \dot{\vec{A}} = A\dot{A}$ or $\vec{A} \cdot d\vec{A} = A\,dA$. Then using these relations we can rewrite Eq. 6 in the following manner,

$$\frac{1}{2}d\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) = -\frac{\mu}{r^2}dr = d\left(\frac{\mu}{r}\right) \qquad \textbf{(7)}$$

Integrating both sides of Eq. 7 leads to

$$\frac{1}{2}\vec{V} \cdot \vec{V} - \frac{\mu}{r} = \frac{V^2}{2} - \frac{\mu}{r} = En \qquad \textbf{(8)}$$

Eq. 8 is the specific energy equation and says that the kinetic energy plus the potential energy (all per unit mass) is a constant. Typically we use the symbols T + U = En, where En is the total mechanical energy/unit mass. Therefore, the potential energy for an inverse square gravitational field is given by $U=-\frac{\mu}{r}$, where the datum is selected such that the potential energy is ZERO at infinity. As a consequence of this reference point, the potential energy of ALL ORBITS IS NEGATIVE! The energy equation for all orbits is given by

$$\frac{V^2}{2} - \frac{\mu}{r} = En \qquad \textbf{(9)}$$

En in Eq. 9 represents the fourth constant of integration (4 down, 2 to go), and is the Energy constant. Any given orbit has a constant energy.

Consequences of Angular Momentum Constant

The fact that the angular momentum is a constant simplifies the problem considerably. The angular momentum being constant means that it is fixed in (inertial) space. One can also note that the velocity vector and the position vector are always perpendicular to the angular momentum vector (by definition of the cross product). Furthermore the position vector goes through the center of attraction (Earth, Sun, or whatever is the primary mass). Note that the r and V vectors form a plane. We can now come to the following conclusions:

1. Any given orbit lies in a plane fixed in space.
2. The fixed orbit plane must pass through the center of attraction.

Reducing the Equations of Motion

Since the orbit lies in a fixed plane, the equations of motion may be simplified by noting that we can now (for the time being) reduce the problem to two dimensions. Hence we can write Eq. 1 in plane coordinates, either x and y as in Eq. 2 (with z = 0), or we can use plane polar coordinates. It turns out more useful to reduce the problem by using plane polar coordinates. Formally, we can use the two of the three angular momentum constants to locate the plane in space (we will do that later). This leaves one of the constants associated with angular momentum and a constant associated with the energy, that we can apply to our reduced equations of motion. As we will see, the new equations will require 4 constants of integration, two of which we already have.

The equations of motion can now be written in plane polar coordinates. We will write the radial acceleration equals the radial force per unit mass, and the transverse acceleration equals the transverse force per unit mass. From previous work we can write directly,

$$\begin{aligned} \hat{e}_r: &\qquad \ddot{r} - r\dot{\theta} = -\frac{\mu}{r^2} \\ \hat{e}_\theta: &\qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{aligned} \tag{10}$$

These equations are two second order, ordinary differential equations in the dependent variables r and θ, with in the independent variable, t. A solution consists of determining r(t) and θ(t). Determining such a solution requires determining 4 constants of integration. Such a solution WILL NOT BE POSSIBLE. However we will try and extract as much information as we can. Remember, two of the constants that we already have must be contained in Eq. 10.

If we look at the transverse (θ) equation, we can note that it can be rewritten as

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{11}$$

Assuming r ≠ 0, Eq. 11 gives us the angular momentum constant, in particular, the magnitude of the specific angular momentum is constant.

$$\boxed{r^2 \dot{\theta} = h = const \quad \text{(magnitude of angular momentum)} \qquad \textbf{(12)}}$$

The resulting equations to be solved are:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu}{r^2} \\ r^2\dot{\theta} &= h \end{aligned} \qquad \textbf{(13)}$$

Here we have a second order and a first order ordinary differential equation, whose solution is r(t) and θ(t), which we are unable to obtain. Our strategy then is to change the independent variable from time, t, to the angle, θ. This procedure is accomplished by noting that we would like to generate the derivative dr/dθ. We can obtain this derivative by the following method,

$$\frac{d(\cdot)}{d\theta} = \frac{\dfrac{d(\cdot)}{dt}}{\dfrac{d\theta}{dt}} = \frac{1}{\dot{\theta}}\frac{d(\cdot)}{dt} = \frac{r^2}{h}\frac{d(\cdot)}{dt} \qquad \textbf{(14)}$$

Therefore we can replace any derivative with respect to time with,

$$\frac{d(\cdot)}{dt} = \frac{h}{r^2}\frac{d(\cdot)}{d\theta} \qquad \textbf{(15)}$$

We can apply Eq. 15 to the radial component of the acceleration found in Eq. 13, and use the transverse equation in Eq. 13 to remove the $\dot{\theta}$ term. The result is,

$$\frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \qquad \textbf{(16)}$$

This equation seems worse than the one we started with! However, we will now seek to look at a change of independent variable to see if we can simplify Eq. 16. The clue we get is from the term inside ( ). We can note that if we let u = 1/r, then

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} \tag{17}$$

Noting that h is a constant and replacing r with 1/u, and $\dot{\theta}$ with h/r^2, and dividing through by h^2u^2 Eq. 16 becomes,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{18}$$

which has a well known solution. The solution is that of a simple harmonic oscillator with a constant forcing function. It can be written in several ways, all equivalent. Here we will introduce two constants, but only one of which is a new constant. The solution to Eq. 18 is given by,

$$\begin{aligned} u(\theta) &= A\cos\theta + B\sin\theta + \frac{\mu}{h^2} \\ &= C\cos(\theta-\omega)+\frac{\mu}{h^2} \\ &= \frac{\mu}{h^2}[1 + e\cos(\theta-\omega)] \end{aligned} \tag{19}$$

where e and ω are the two constants. However only ω is a new one. (5 down, 1 to go). We will show that e is related to the magnitude of the angular momentum and to energy. We can now note that since u = 1/r we can just take the reciprocal of Eq. 19 to get a general expression for r(θ),

$$\boxed{\begin{aligned} r(\theta) &= \frac{\frac{h^2}{\mu}}{1 + e\cos(\theta - \omega)} \\ &= \frac{\frac{h^2}{\mu}}{1 + e\cos\nu} \end{aligned} \tag{20}}$$

where θ - ω has been replace by ν.

Eq. 20 is the equation of a CONIC SECTION where r is measured from the focus. The quantity ν is called the TRUE ANOMALY and is the angle measured from the radius vector defined by the point of the conic section closest to the origin, and the current position (radius) vector. The quantity e is called the eccentricity of the conic section.

A quantity called the orbit parameter p is defined such that

$$r = \frac{p}{1 + e \cos \nu} \tag{21}$$

The orbit parameter $p = h^2/\mu$. It is also called the ***semi-latus rectum***, and is the radius distance when the true anomaly ν= 90 deg ($\pi/2$ rad). Eqs. 20 and 21 are the solution to the equations of motion when time is eliminated. They are called the equation of the orbit and describe how the radial distance varies with the change in the central angle. The particular angle ν is called the true anomaly and is measured from the point of closest approach, i.e. $r = r_{min}$ when $\nu = 0$. Further, the solution is the equation of a conic section with the radius vector measured from a focus of that conic section. Consequently from the properties of conic sections, we know that,

$e < 1$, conic section is an ellipse
$e = 1$, conic section is a parabola
$e > 1$, conic section is an hyperbola

In both parabolic and hyperbolic orbits the radius vector becomes infinitely long and they represent orbits on which the satellite escapes the two-body system.