Vectors and Their Representations

A vector is an abstract quantity introduced by mathematicians and engineers to represent a quantity which has magnitude and direction. Typically in this course we are interested in three dimensional vectors representing position, velocity, and acceleration. A vector is generally represented in its abstract form by an arrow whose length is proportional to the magnitude of the vector quantity. We can deal directly with the vector quantity when doing certain operations such as addition, subtraction, vector, and scalar multiplication by using graphical techniques and by just indicating them, $\vec{A} + \vec{B} = \vec{C}$. However, to deal with calculations it is generally necessary to use the REPRESENTATION of the vector. These representations usually require one to define a coordinate system. In general the same vector can be represented in many different ways, depending upon the coordinate system selected. We can pick different types of coordinate systems, e.g. rectangular, spherical, plane polar, torroidal, elliptical, etc. and we can pick different orientations of the same type of system. In either case, the representation of the same vector will appear quite different. Although in general we are usually interested in different orientations of the same type, at the present time we are interested in different types, in this case rectangular and plane polar coordinate systems.

We generally define a coordinate system by a set of mutually orthogonal unit vectors, called basis vectors. These vectors are of unit length and are perpendicular to each other forming a unit triad. Typically the are designated by the symbol $\hat{e}_i$, where I indicates a direction. For example the following is an equivalent representation of the generic vector $\vec{A}$ in a rectangular coordinate system:

$$\begin{aligned} \vec{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \\ &= A_x \hat{e}_x + A_y \hat{e}_y + A_z e_{\hat{z}} \\ &= A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \end{aligned} \tag{1}$$

or

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i \tag{2}$$

Here, the A_i terms are called components of the vector and the $\hat{i}, \hat{j}, \hat{k}, \hat{e}_x, \hat{e}_y, \hat{e}_z, (\hat{e}_i,\ i = 1,2,3)$ are the basis vectors for this coordinate system.

The position vector in a rectangular coordinate system is generally represented as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{3}$$

with $\hat{i}, \hat{j}, \hat{k}$ being the mutually orthogonal unit vectors along the x, y, and z axes respectively. The values x, y, and z are the ***scalar*** components of the position vector $\vec{r}$.

All coordinate systems have two items in common, a reference plane, and a direction in that reference plane. For rectangular coordinates one can think of the reference plane as the x-y plane and the reference direction as the x direction. Further, the coordiante z is measured perpendicular to the reference plane, giving us the coordinates (x, y, z). If we consider plane polar (or cylindrical) coordinates, the reference plane is the one in which the radial component is measured, (r), and the reference direction, the one from which the angle to the radial component is measured (θ). In addition, in cylindrical coordinates, the coordinate z is measured perpendicular to the reference plane, giving us the coordinates (r, θ, z). In spherical coordinates we can think of some equatorial-like plane as the reference plane. The magnitude of the position vector (r) is one coordinate. The reference direction is that direction from which the angle to the projection of the position vector on the reference plane is measured (θ), and the elevation of the position vector with respect to the reference plane is the third coordinate (ϕ), giving us the coordinates (r, θ, ϕ).

Here, for reasons to become clear later, we are interested in plane polar (or cylindrical) coordinates and spherical coordinates. Cylindrical coordinates have mutually orthogonal unit vectors in the radial (parallel to the radius vector), transverse (perpendicular to the radius vector in the plane of interest) and normal (perpendicular to the plane of interest). They are designated as $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ respectively. A generic vector $\vec{A}$ would be represented as:

$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z, \tag{4}$$

where A_r, A_θ, A_z are the scalar radial, transverse, and z components of the vector $\vec{A}$.

Spherical coordinates also have mutually orthogonal unit vectors in the radial (parallel to the position vector), the longitudinal (parallel to the reference plane and perpendicular to the position vector), and the elevation or latitude (along a constant longitude line and perpendicular to the position and longitudinal unit vectors). A generic vector $\vec{A}$ would be represented as:

$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\phi \hat{e}_\phi, \tag{5}$$

where A_r, A_θ, A_ϕ are the scalar radial, longitudinal, and latitudinal components of the vector $\vec{A}$.

It should be clear that the scalar components of the representation of the vector $\vec{A}$ in plane polar or spherical coordinates are not the same as those in rectangular coordinates. Hence the same vector has a different representation in different types of coordinate systems. Also it should be clear that the same vector will have a different representation in two rectangular coordinate systems oriented in different directions.

Although the two representations are different in the two systems, they are related to each other. If we consider the same vector represented in a rectangular coordinate system and in a plane polar coordinate system, we have the following relations between the two representations:

$$
\begin{aligned}
\hat{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \\
&= A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z, \qquad (6) \\
&= A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\phi \hat{e}_\phi,
\end{aligned}
$$

where even the components A_r and A_θ are different in the two different representations.

The relationship amongst the various components is called a transformation. We can write the transformation matrix relating the cylindrical and spherical components of the vector to the rectangular components. The results are for rectangular to cylindrical:

$$
\begin{Bmatrix} A_r \\ A_\theta \\ A_z \end{Bmatrix}^{cyl} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}^{rect}, \qquad (7)
$$

and for spherical:

$$
\begin{Bmatrix} A_r \\ A_\theta \\ A_\phi \end{Bmatrix}^{sphere} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}^{rect} \qquad (8)
$$

Hence, although the representations of the same vector are different in different coordinate systems, these representations are generally related to each other. Note that since these transformation matrices are orthogonal matrices, their inverse can be obtained by just taking the matrix transpose.

VECTOR ALGEBRA

Addition and Subtraction

Generally we can manipulate vector equations using the generic vector itself. However, it is useful to know how to the basic vector algebra using the representations of the vector. For addition and subtraction we have for the generic operation,

$$\vec{A} + \vec{B} = \vec{C} \tag{9}$$

We can actually perform this operation in terms of the vector representations by noting that we must have each vector represented in the same coordinate system. Then the addition and subtraction operation is just the addition and subtraction of the vector components.

Scalar Product

The scalar (dot) product is generically given as $\vec{A} \cdot \vec{B} = S$ where S is a scalar (independent of coordinate system in which $\vec{A}$ and $\vec{B}$ are written). Furthermore, the definition is $\vec{A} \cdot \vec{B} = |\vec{A}|\ |\vec{B}|\ \cos\angle(\vec{A},\vec{B})$. In terms of the vector representations, one can use the definition of the scalar product to show it can be calculated simply as the sum of the products of like components. Here we have

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i\ , \tag{10}$$

or

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z, \\ &= A_r B_r + A_\theta B_\theta + A_z B_z, \\ &= A_r B_r + A_\theta B_\theta + A_\phi B_\phi, \end{aligned} \tag{11}$$

and the result is the same scalar, regardless of which representation is used.

The magnitude of a vector is given by:

$$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} \tag{12}$$

Vector Product

The vector product is generically given as $\vec{A}\, x\, \vec{B} = |\vec{A}|\ |\vec{B}|\ \sin\angle(\vec{A},\vec{B})\hat{e}_\perp$, where $\hat{e}_\perp$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and directed in accordance with the right hand rule. If you rotate $\vec{A}$ into $\vec{B}$ then the vector $\hat{e}_\perp$ points in the direction in which a right handed screw would advance. Alternatively, take your right hand, point you fingers in the direction of $\vec{A}$ and curl them towards $\vec{B}$ and your thumb will be pointing in the direction of $\hat{e}_\perp$. This same operation can be performed using the representations of the vector in the following manner.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{Bmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{Bmatrix}, \tag{13}$$

or

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ A_R & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix} = \begin{Bmatrix} A_\theta B_z - A_z B_\theta \\ A_z B_r - A_r B_z \\ A_r B_\theta - A_\theta B_r \end{Bmatrix}, \tag{14}$$

or

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ A_R & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix} = \begin{Bmatrix} A_\theta B_\phi - A_\phi B_\theta \\ A_\phi B_r - A_r B_\phi \\ A_r B_\theta - A_\theta B_r \end{Bmatrix}. \tag{15}$$

Vector Calculus

We generally have to deal with derivatives of the above vectors. Or particular interest here is the representation in plane polar coordinates. It should be noted that the basis vectors in the (inertial) rectangular coordinate system do not change in magnitude or direction, and hence are constant. In plane polar coordinates, the basis vectors are constant in magnitude, but are changing direction. Hence their derivatives are not zero. We can note the following development:

$$\dot{\hat{e}}_r = \lim_{\Delta t \Rightarrow 0} \frac{\Delta \hat{e}_r}{\Delta t} = \lim_{\Delta t \Rightarrow 0} \frac{\Delta\theta\, \hat{e}_\theta}{\Delta t} = \dot{\theta}\, \hat{e}_\theta. \tag{16}$$

Similarly,

$$\dot{\hat{e}}_\theta = -\dot{\theta}\, \hat{e}_r. \tag{17}$$

Then, since $\vec{r} = r\,\hat{e}_r$,

$$\frac{d\vec{r}}{dt} = \vec{V} = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r = \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta = V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta. \tag{18}$$

In a similar manner we can represent the acceleration.

$$\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{V}}{dt} = \ddot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + r\,\ddot{\theta}\,\hat{e}_\theta + r\,\dot{\theta}\,\dot{\hat{e}}_\theta,$$

or

$$\begin{aligned}\vec{a} &= \frac{d\vec{V}}{dt} = \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + r\,\ddot{\theta}\,\hat{e}_\theta - r\,\dot{\theta}^2\,\hat{e}_r \\ &= (\ddot{r} - r\,\dot{\theta}^2)\,\hat{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\hat{e}_r\end{aligned} \tag{19}$$

Similar, but much more complicated, calculations can be carried out for spherical coordinates. The resulting unit vector rates can be determined to be:

$$\begin{aligned}\dot{\hat{e}}_r &= \dot{\phi}\,\hat{e}_\phi + \dot{\theta}\cos\phi\,\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= \dot{\theta}\sin\phi\,\hat{e}_\phi - \dot{\theta}\cos\phi\,\hat{e}_r \\ \dot{\hat{e}}_\phi &= -\dot{\phi}\,\hat{e}_r + \dot{\theta}\sin\phi\,\hat{e}_\theta\end{aligned} \tag{20}$$

Summary

The position, velocity, and acceleration for each coordinate system are given next.

Rectangular Coordinates

$$\begin{aligned}\vec{r} &= x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \\ \vec{V} &= \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k} \\ \vec{a} &= \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k}\end{aligned}$$

Polar coordinates (in-plane components only)

$$\begin{aligned}\vec{r} &= r\,\hat{e}_r \\ \vec{V} &= \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta \\ \vec{a} &= (\ddot{r} - r\,\dot{\theta}^2)\,\hat{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\hat{e}_\theta\end{aligned}$$

(21)

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$
$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial x}\hat{i} + \frac{\partial(\cdot)}{\partial y}\hat{j} + \frac{\partial(\cdot)}{\partial z}\hat{k}$$
$$\nabla\Phi\cdot d\vec{r} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

(22)

$$\vec{F} = F_e\,\hat{e}_r + F_\theta\,\hat{e}_\theta$$
$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$
$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial(\cdot)}{\partial\theta}\hat{e}_\theta$$
$$\nabla\Phi\cdot d\vec{r} = \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta$$
$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Spherical Coordinates:

$$\vec{r} = r\,e_r$$

$$\begin{aligned}\vec{V} &= V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_\phi\hat{e}_\phi \\ &= \dot{r}\,\hat{e}_r + r\dot{\theta}\cos\phi\,\hat{e}_\theta + r\dot{\phi}\,\hat{e}_\phi\end{aligned} \tag{24}$$

$$\begin{aligned}\vec{a} &= a_r\hat{e}_r + a_\theta\hat{e}_\theta + a_\phi\hat{e}_\phi \\ &= (\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\cos^2\phi)\hat{e}_r + (r\ddot{\theta}\cos\phi + 2\dot{r}\dot{\theta}\cos\phi - 2r\dot{\phi}\dot{\theta}\sin\phi)\,\hat{e}_\theta \\ &\qquad + (r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\dot{\theta}^2\sin\phi\cos\phi)\,\hat{e}_\phi\end{aligned}$$

$$\vec{F} = F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_\phi\hat{e}_\phi$$
$$d\vec{r} = dr\,\hat{e}_r + r\cos\phi\,d\theta\,\hat{e}_\theta + r\,d\phi\,\hat{e}_\phi$$
$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial r}\hat{e}_r + \frac{1}{r\cos\phi}\frac{\partial(\cdot)}{\partial\theta}\hat{e}_\theta + \frac{1}{r}\frac{\partial(\cdot)}{\partial\phi}\hat{e}_\phi$$
$$\nabla\Phi\cdot d\vec{r} = \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta + \frac{\partial\Phi}{\partial\phi}d\phi$$

$$\begin{aligned}x &= r\cos\phi\cos\theta \\ y &= r\cos\phi\sin\theta \\ z &= r\sin\phi\end{aligned} \tag{25}$$

Finally we note:

$$\vec{A}\cdot\dot{\vec{A}} = A\dot{A} \qquad \text{and} \qquad \vec{r}\cdot\dot{\vec{r}} = r\dot{r}$$