C9 "*e*" or here *E* in aerodynamics. W.H. Mason, Oct. 28, 2015/July 11, 2017

An airplane's subsonic drag polar is often described in terms of the basic parasite drag, C_{D0}, and a drag due to lift term typically given as $C_L^2/(\pi ARE)$. This can be more complicated than it appears. Apparently the nomenclature can vary with each company's internal terminology. At Grumman I was taught that the famous Glauert analysis of the spanload shape was referred to with a lower case *e*. The overall airplane drag polar was described using an upper case *E*. As we'll see, the two differ. Apparently this terminology was unique to Grumman. It seems everybody else uses a lower case *e* (the lower case *e* was used by Oswald for the entire airplane in his NACA Report 408 in 1931). I'll stick with the Grumman notation, it's too confusing otherwise. Also, although students like to think of this value as a constant, it's not. Performance calculations normally require as input a table of $E(C_L, M)$. This table comes from the aerodynamicist.

This curiosity is intended to add some perspective, mainly for students, and extends a lot of the discussion from Curiosity C8. I was troubled by the way the value of *E* is developed in typical aerodynamics textbooks and the values of *E* suggested in aircraft design books. The values suggested in aircraft design texts are much lower than those that would be found from typical analysis contained in aerodynamics books.[1]

Review for Unswept Wings

One of the most important concepts in aerodynamics is Prandtl's Lifting Line Theory (LLT). It describes the downwash distribution across the wing arising from the trailing vortex system. This means that airfoils at different span stations "see" an angle of attack that differs from the apparent geometrical angle of attack by an induced angle, rotating the force normal to the apparent velocity the airfoil sees aft, resulting in an induced drag component of the force (remember that the drag from inviscid flow theory of an airfoil in two dimensional flow is zero). Prandtl's LLT is an integral equation for the distribution of the value of the downwash, which provides the change in angle of attack that the airfoils see at each spanwise location. Glauert's elegant solution of the integral equation in terms of a Fourier series is given in textbooks to find the induced drag arising from the downwash distribution. The elliptic distribution of the spanload as the one that minimizes the induced drag follows directly from his solution, as well as providing a means for determining the additional induced drag when the spanload is not elliptic.

The Glauert solution, which is for planar unswept wings, has been historically given as

$$C_{D_i} = \frac{C_L^2}{\pi AR}\left(1+\delta\right)$$

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[1] Realize that in early design the aerodynamicist needs to provide an "aggressive" value of *E*. Detailed aerodynamic design is in large part about achieving a low C_{D0} and a high *E*. Too low an *E* and the competition will win the job. At the same time evaluators will be scrutinizing the value of *E* to decide if it's realistic. And if you are lucky the design will be built and then the aero guy will be held accountable for his estimate of *E*!

and this is normally rewritten as $C_{D_i} = \frac{C_L^2}{\pi ARe}$, where $e = \frac{1}{1+\delta}$. For planar wings e is always less than or equal to one.[2]

As we saw in Curiosity C8, the airfoil drag has to be added to the induced drag to obtain the complete wing drag. We can use the drag due to lift component to find E. Start with the complete drag:

$$C_D = \int_0^1 \frac{cc_d(c_l)}{c_a} d\eta + C_{D_i} = C_{D_0} + \int_0^1 \frac{c\Delta c_d(c_l)}{c_a} d\eta + \frac{C_L^2}{\pi AR\mathrm{e}}$$

As an example we'll simplify this by picking the elliptic planform we used in Curiosity C8 and the NACA 0012 airfoil. This makes the calculation easy because the section c_l is constant across the wing and hence Δc_d is constant across the span. In addition, we found that a parabola was a good fit to the airfoil drag. Thus the drag due to lift is (recall that for an elliptic planform the section c_l and wing C_L are equal):

$$C_{D_L} = ac_l^2 + \frac{C_L^2}{\pi AR}, \quad \text{or } C_{D_L} = \frac{C_L^2}{\pi ARE}, \qquad \text{where } E = \frac{1}{1+\pi ARa}$$

where we found $a = 0.004666$ for the NACA 0012 airfoil data in Abbott and von Doenhoff. This can be considered the "ultimate" wing value of the Oswald E for this airfoil. The Glauert span efficiency factor is one in this case and E is not a function of C_L. Lets see what the values of E are for aspect ratios of 6 and 20. For an AR of 6, $E = 0.919$, while for an AR of 20, $E = 0.773$. This supports the comment by Steve Brandt that the airfoil performance becomes important as the aspect ratio increases (and hence the induced drag contribution to drag due to lift decreases).

In general we'll have to do the integral for the extra drag numerically. This is done by extending my lifting line theory program to include the extra airfoil drag. I checked this extension fairly carefully. The value of E is now found from:

$$E = \frac{C_L^2}{\pi AR\left(C_{D_i} + \Delta C_{D_{foil}}\right)}$$

Now we will look at some values of E for unswept wings. We compare the relatively simple formulas proposed for design students and the actual values of airplanes found in the literature (E is not something the airframers like to publish!). In particular Steve Brandt at USAFA sent me info for a revised E to be included in the 3rd Ed. of his book. We will use that and the formulas given by Raymer as a baseline for our curiosity. Brandt's formula shows up in AIAA 2015-2486 (with a minor typo). I currently suggest that our students use this:

$$e = 4.61(1 - 0.033AR^{0.53})\cos^{0.1}\Lambda_{LE} - 3.3$$

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[2] The value of e can be found for a planar wing with a specified spanload from the LIDRAG program on my software website. Alternately, the lifting line theory program on the site can also be used for unswept wings.

The unswept wing estimates from Brandt, Raymer and the values of E that arise from adding the airfoil drag to the induced drag in Curiosity 8 and from the modification of my lifting line theory program for a straight tapered wing ($\lambda = 0.8$) are shown and in Figure 1 over a range of aspect ratios of from 4 to 22 (Dan Raymer says explicitly not to use his formula for high aspect ratios, *i.e.*, sailplanes). Note that the Curiosity 8 work led to an analytic value for an uncambered airfoil on an elliptic planform unswept wing. The formula turns out to be

$$E = \frac{1}{1+\pi ARa}$$

where $\Delta c_d = ac_l^2$, and for the NACA 0012 airfoil in Figure 1 of Curiosity 8, $a = 0.004666$.

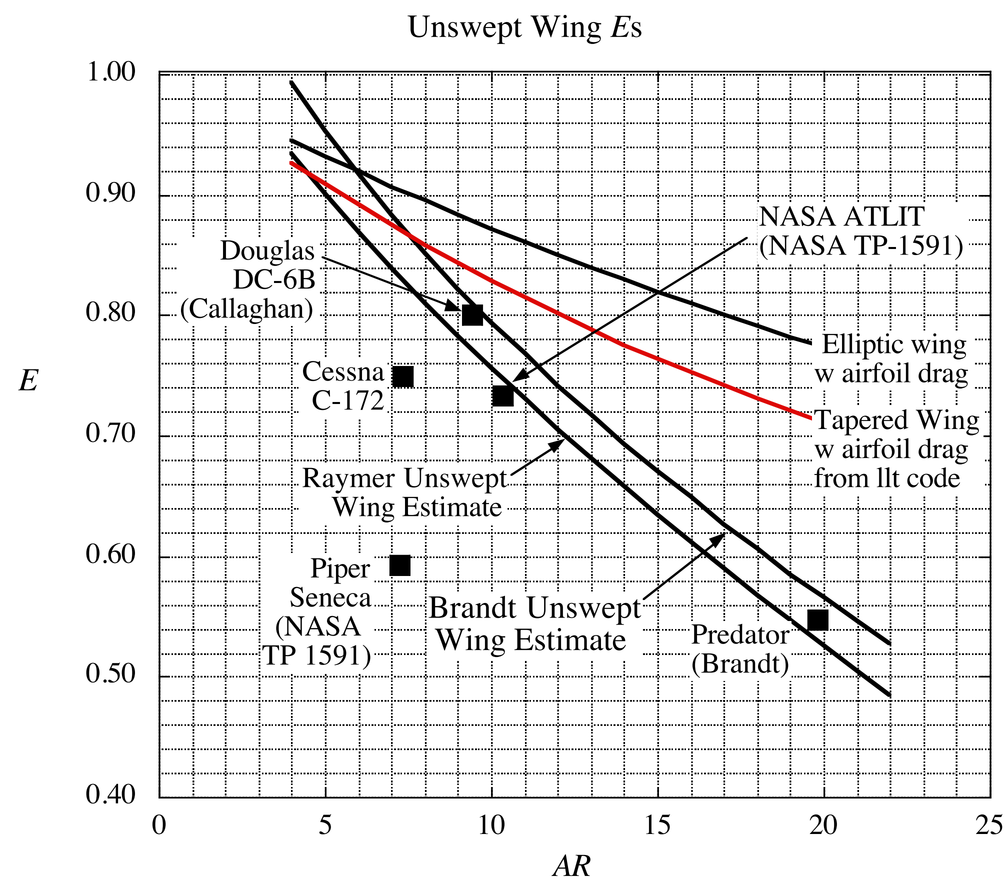

Figure 1. Brandt's E estimate compared with Raymer's Eqn 12.49 of the 4th Ed. and the value from the work in Curiosity 8 for an unswept wing and for a modified lifting line estimate.

The first thing we notice in Fig. 1 is the wide variation of the values of Es from real aircraft. The trends from the estimates by Brandt and Raymer are similar. The "ultimate" elliptic wing estimate predicts higher Es than Brandt or Raymer and as expected there is a reduction from that value when we solve for E using the lifting line theory with added airfoil drag. It's curious that the slopes with respect to aspect ratio are so different than the Brandt and Raymer estimates. The value of E for the Predator validates Brandt's equation. I'm at a loss for an explanation of exactly why the value is so low. The NASA

ATLIT is said to suffer from a poor spanload shape due to the nacelles on the wing, and they say this could be addressed to improve *E*.

Next we will look at some values of *E*s from real airplanes that I've found in the literature and my personal data collection. Figure 2 and 3 are from my personal collection. The important observation here is that *E* is a function of the lift coefficient.

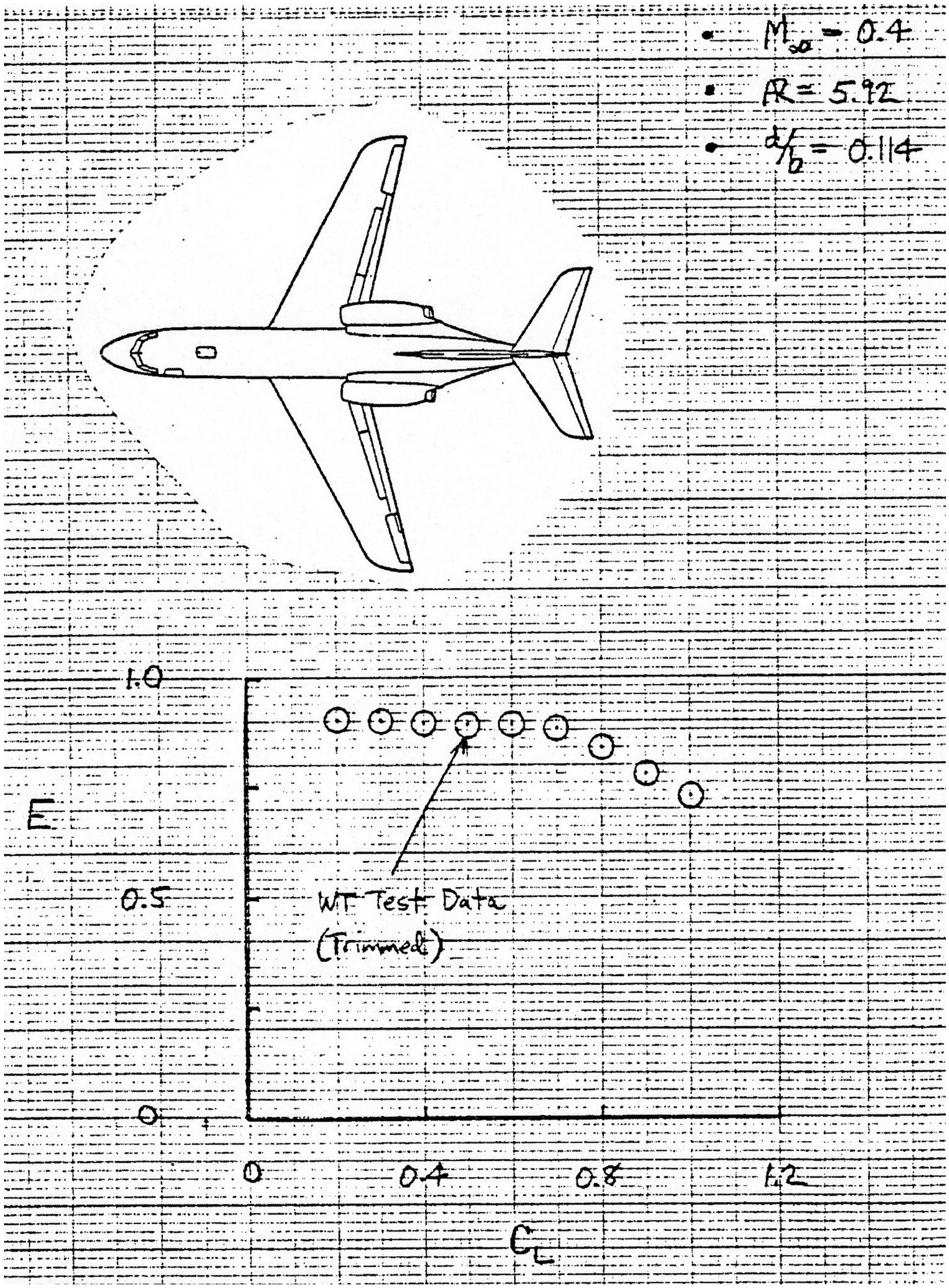

Figure 2. A typical business jet.

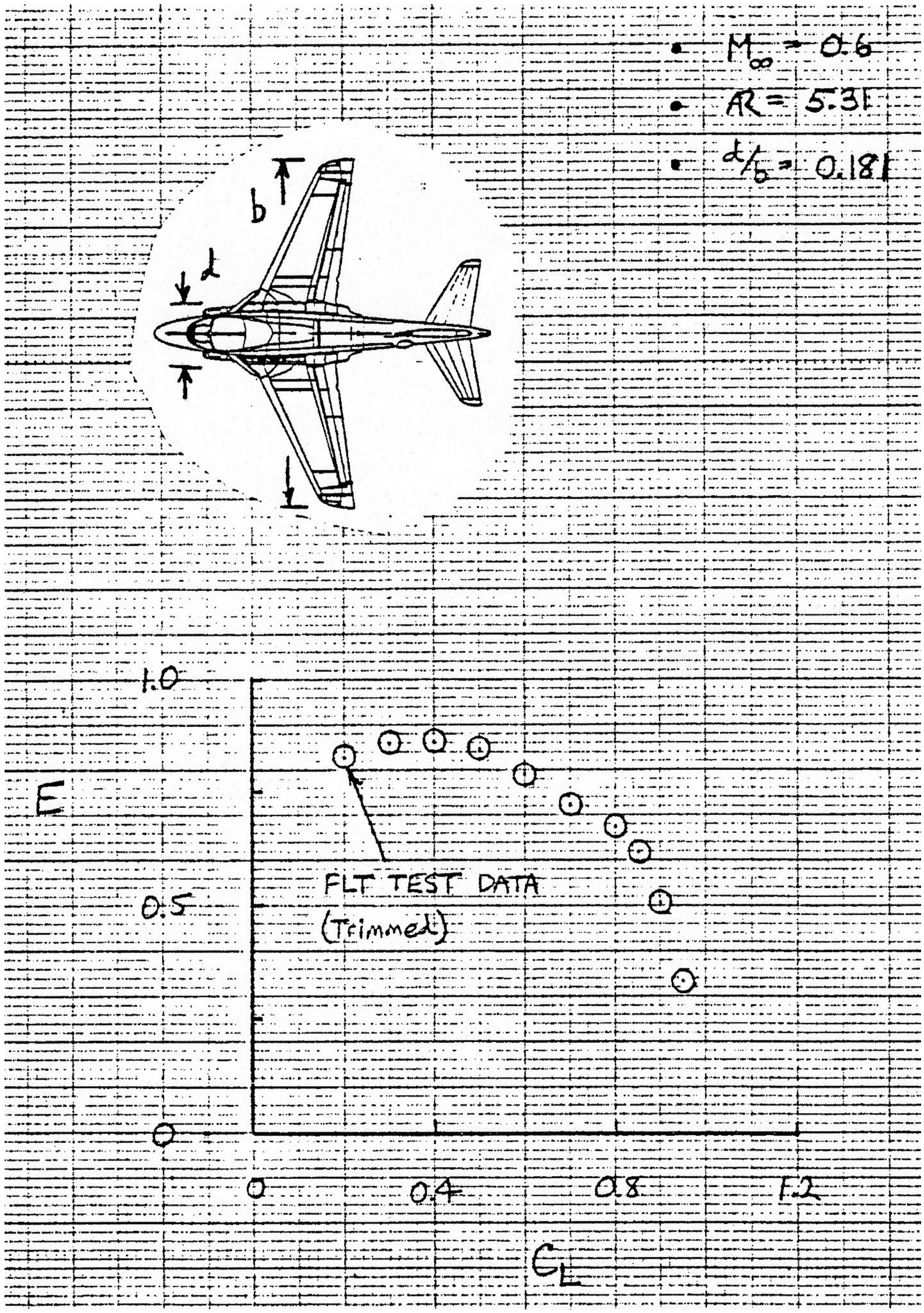

Figure 3. A typical tactical aircraft.

We can also look at a case with data available from a NASA wind tunnel test. NASA TN D-5805 provides an extensive amount of tab data, making it easy to find E. The report was primarily to investigate buffet and has a wealth of tab data. To find *E*. Simply find:

$$E = \frac{C_L^2}{\pi AR(C_D - C_{D_0})}$$

Figure 4 shows the planform, and Figure 5 has the resulting *E* values from the wind tunnel test. This is for an untwisted, uncambered wing, so it is not a good aero design. The *E* is as high as 0.74. The Figure shows how E varies with C_L, with the highest values of *E* occurring at C_Ls that would be around the cruise value.

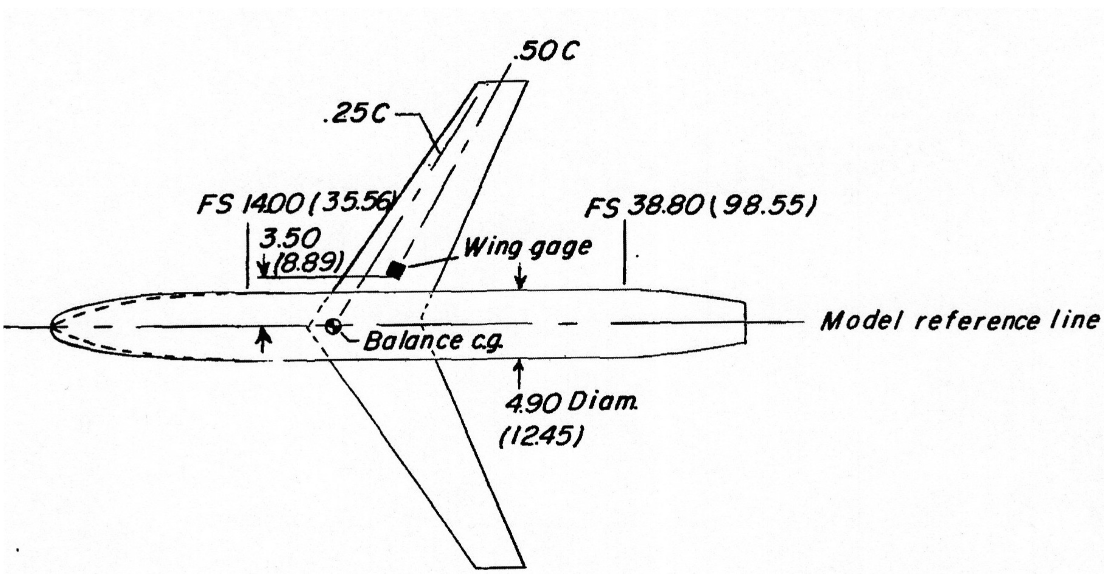

Figure 4. Wind Tunnel Model Planform (NASA TN D-5805)

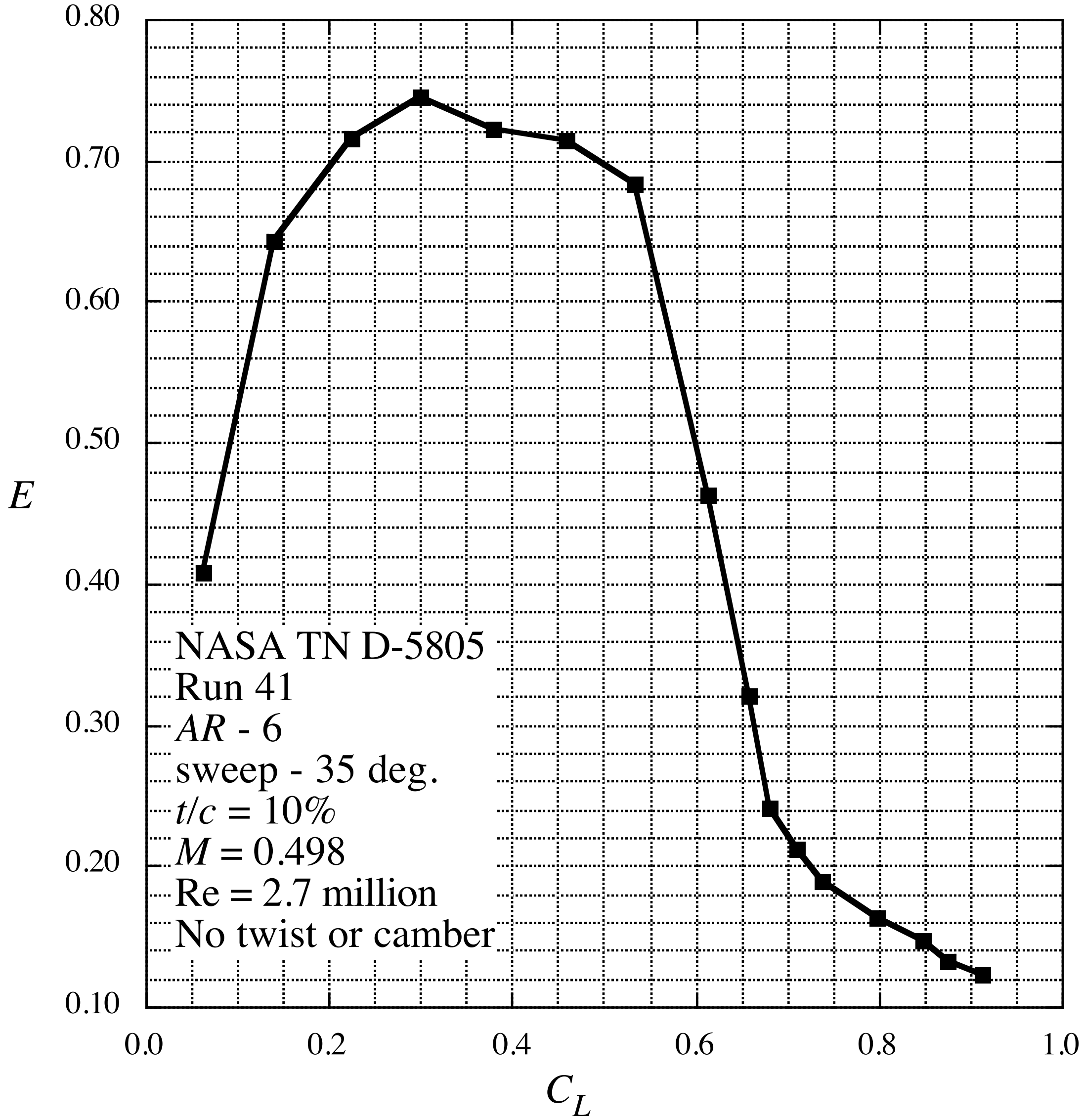

Figure 5. E extracted from wind tunnel test tab data(NASA TN D-5805)

We complete our examination of values of E using data from NASA TP 3414. That report had an X-29 assessment, including a comparison with other fighters. Figure 6 shows the X-29 flight determined values of E for several Mach numbers. The solid symbols are the values found in flight. At supersonic speeds it is appropriate to use K to describe the drag due to lift performance. The subsonic/transonic E values are around 0.9, and decrease as the Mach number approaches one and transonic effects start to impact the E. Recall that the X-29, as do all fighter type planes, has a low aspect ratio. Above in Figure 1 we saw the trend toward higher Es at low aspect ratios. Nevertheless, the X-29 was primarily designed to attain efficient drag due to lift and it appears to have succeeded.

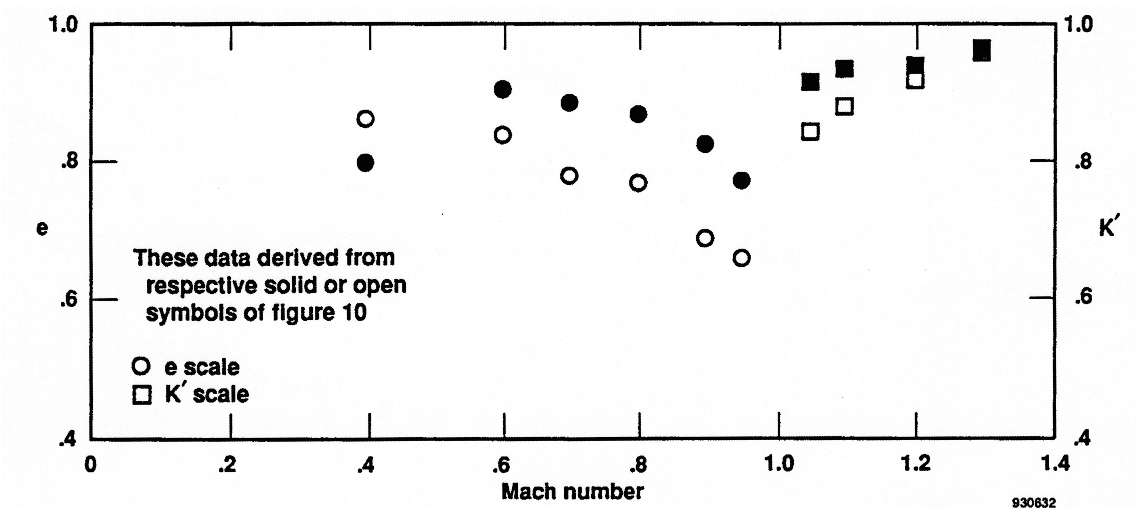

Figure 6. X-29 values of drag due to lift from NASA TP 3414
(solid symbols are from flight test)

Figure 7 shows the values of *E* for the X-29 compared to other fighters. It has a higher *E* than any of the others, although sometimes just slightly.

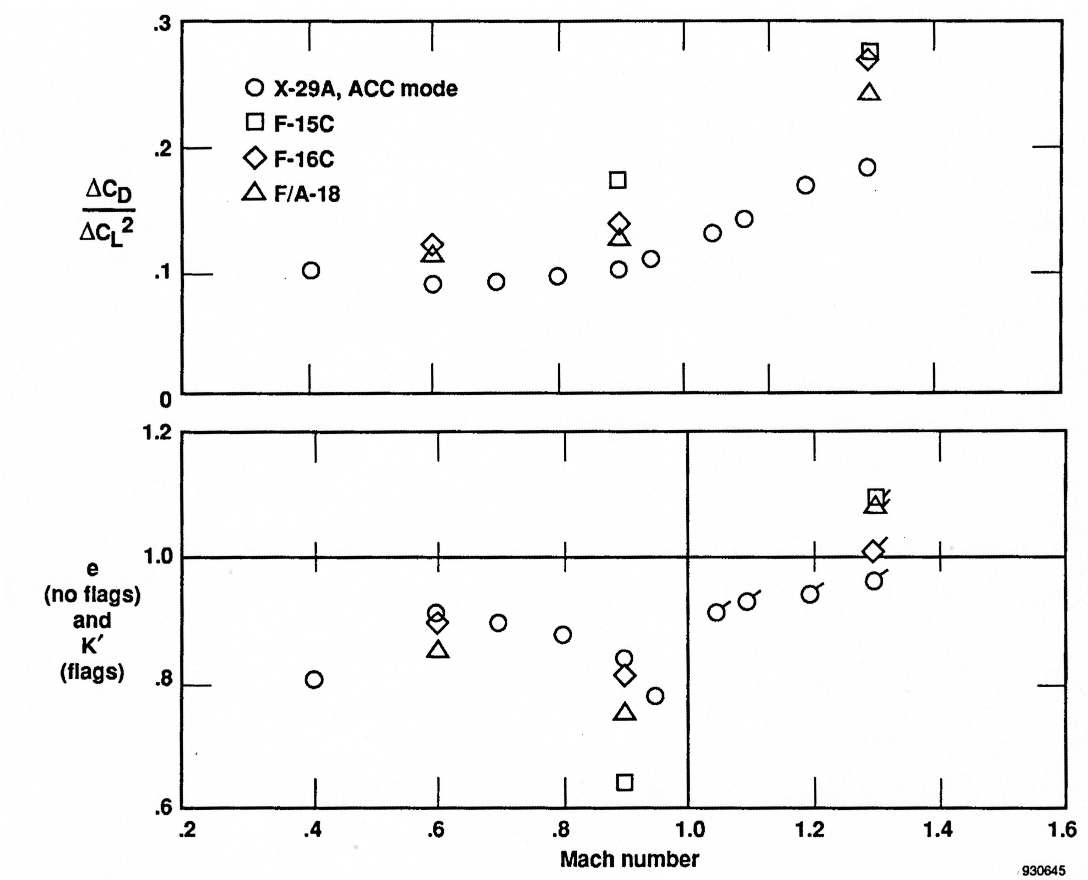

Figure 7. X-39 drag due to lift compared to other fighters (NASA TP 3414).

Some takeaways:

- There's a big difference between the spanload efficiency factor *e* and the Oswald *E*.
- There's big variation in *E*s demonstrated by various airplanes. Many fine details of the aerodynamics can affect E.
- The nomenclature for drag due to lift varies wildly, you have to pay attention. For example, look at Shevell's text and the book by Abbott and von Doenhoff.
- Although we primarily looked at unswept wings, the extension of the spanload concept to the Trefftz plane means that the spanload eddiciency is also valid for swept wings. The Mach number only enters as it affects the shape of the spanload.
- To make things more complicated, you can include the fuselage in the calculation by reducing *E* by a factor. This can be found in a chart in Shevell or by using an equation in Nicolai (Grumman analysis tended to validate Nicolai's equation):

  $E_{wing-body} = E_{wing}\left[1-\left(\frac{d}{b}\right)^2\right]$, where *d* is the fuselage diameter and *b* is the wing span.

- There are many alternate approaches. One other standard approach is to use the concept of percent leading edge suction, a story left for another day.
- "Camber drag" results in a shift of the drag polar. This arises due to both wing twist and airfoil camber. It complicates the details of the story. Fighters have used leading- and trailing-edge flap schedules for decades and it appears that this is now being done on commercial transports. This variable camber (possibly called *morphing*) is another topic for a separate discussion.

Some useful references

M. Nita and D. Scholz, "Estimating the Oswald Factor from basic aircraft geometrical parameters," Hamburg University of Applied Sciences Aero – Aircraft Design and Systems Group, Berliner Tor 9, 20099, Hamburg, Germany, Deutscher Luft- und Raumfahrtkongress 2012, DocumentID: 281424. When I started putting my notes together for this curiosity I was searching the web and found this paper. It's comprehensive, includes nonplanar configurations (even using Joel's iDrag code from my website) and I really couldn't do better. I highly recommend it if you want a deeper dive into estimating Oswald's *E*. It also has tables of *E*'s for many airplanes.

J. G. Callaghan, "Aerodynamic Prediction Method for Aircraft at Low Speeds with Mechanical High Lift Systemsa," AGARD–LS-67, 1976. *Prediction Methods for Aircraft Aerodynamic Characteristics*. Callaghan was at Douglas Aircraft and since that's where Oswald worked, he uses a lower case *e* for what we are calling capital *E*. The title is misleading, he has a brief section on clean wing *E* and a figure with *E*'s from Douglas Aircraft. I don't recall what reference sent me to find this paper.

Edwin J. Saltzman and John W. Hicks, "In-Flight Lift-Drag Characteristics for a Forward-Swept Wing Aircraft (and Comparisons with Contemporary Aircraft)," NASA TP 3414, 1994. This is the paper that provided the *E* for the X-29 as well as the F-15C, F-16C and F/A-18.