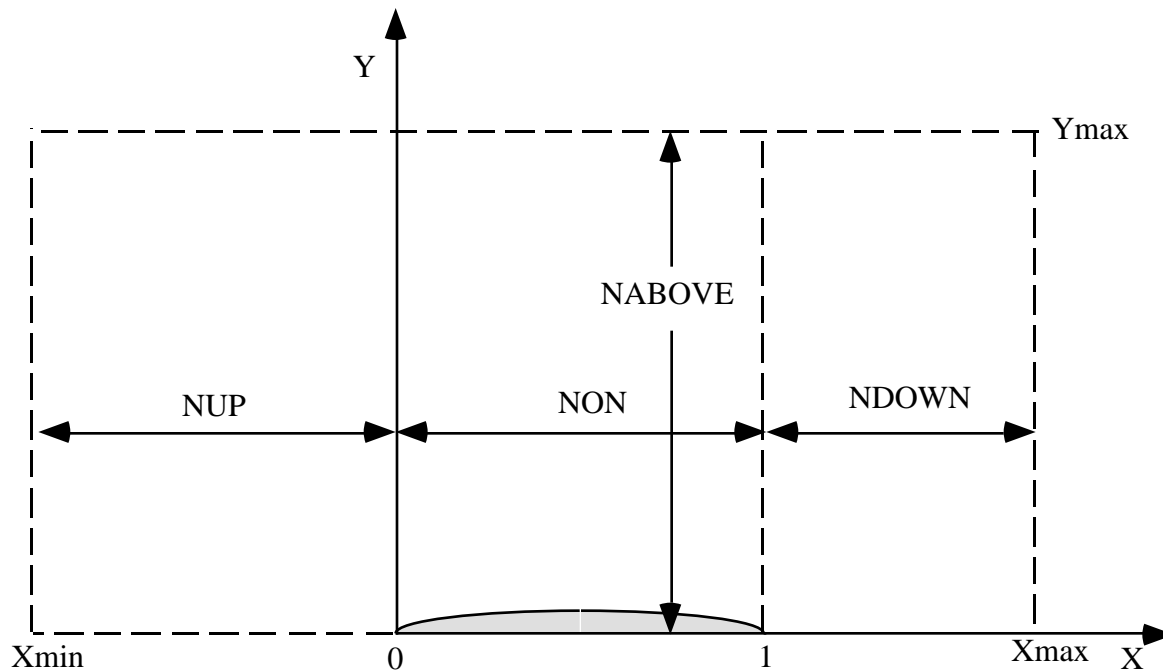


G-1 Program THINFOIL

THINFOIL solves Laplace's Equation by finite differences using a variety of iteration methods. It was written by Valery Razgonyaev. The iteration options include SOR, SLOR, AF1 and AF2 to solve the algebraic equations. An unevenly spaced grid is used to concentrate grid points near the airfoil surface, and near the leading and trailing edges of the airfoil. Notes describing the detailed procedures and program terminology are provided after the sample terminal session.

The definitions of the input values are given in the following figure. Make sure that the value of $Xmin$ is negative, and that the order of input requested by the program is followed. Note that the grid dimensions are real numbers and the number of grid points are integers. Some compilers are very strict about this. These errors are the source of most problems encountered running the code. Note that the airfoil is defined from $x = 0$ to $x = 1$ on $y = 0$. At the farfield boundaries, the potential is set to zero, while on the plane of symmetry, $y = 0$, the normal velocity is set to zero or to the value given by the thin airfoil theory boundary conditions. Running the program, $maxit$ is the number of iterations requested, and RELAX is the value of the over-relaxation factor, ω .



The maximum number of x grid points is 90, however, the program automatically adds points around the boundary, so the user is prevented from exceeding 86. Similarly, the maximum number of y points is 40, but the user is limited to requesting 36. The grid is output, together with the difference between grid points. This is done to see how the grid spacing is changing. The convergence criteria is set internally. In addition, the iteration history is output every three iterations. Study the code to find out how to change this value.

The program creates a copy of the results sent to the screen on a disk file, THINAIR.OUT. This file can then be used to make plots of the results or the convergence history.

Sample terminal session:

```
Program THINFOIL
```

```
Finite difference solution of
the Laplace Eqn. over a biconvex airfoil
```

```
INPUT grid boundary locations; Xmin, Xmax, Ymax:-2.6,3.6,2.6
```

```
INPUT number of points upstream of airfoil,      Nup:18
downstream of airfoil, Ndown:18
on the airfoil, NON:38
above the airfoil, NABOVE:24
```

```
X GRID and slope BC, fp
```

i	x(i)	x(i)-x(i-1)	fp(i)
1	-0.28266E+01		0.00000E+00
2	-0.26000E+01	0.22660E+00	0.00000E+00
3	-0.23734E+01	0.22660E+00	0.00000E+00
4	-0.21485E+01	0.22488E+00	0.00000E+00
5	-0.19271E+01	0.22144E+00	0.00000E+00
6	-0.17107E+01	0.21632E+00	0.00000E+00
7	-0.15012E+01	0.20956E+00	0.00000E+00
8	-0.13000E+01	0.20119E+00	0.00000E+00
9	-0.11087E+01	0.19130E+00	0.00000E+00
10	-0.92875E+00	0.17995E+00	0.00000E+00
11	-0.76152E+00	0.16723E+00	0.00000E+00
12	-0.60828E+00	0.15324E+00	0.00000E+00
13	-0.47020E+00	0.13808E+00	0.00000E+00
14	-0.34833E+00	0.12187E+00	0.00000E+00
15	-0.24360E+00	0.10473E+00	0.00000E+00
16	-0.15680E+00	0.86801E-01	0.00000E+00
17	-0.88593E-01	0.68206E-01	0.00000E+00
18	-0.39500E-01	0.49093E-01	0.00000E+00
19	-0.98938E-02	0.29606E-01	0.00000E+00
20	0.00000E+00	0.98938E-02	0.10000E+00
21	0.17078E-02	0.17078E-02	0.99658E-01
22	0.68193E-02	0.51116E-02	0.98636E-01
23	0.15300E-01	0.84805E-02	0.96940E-01
24	0.27091E-01	0.11792E-01	0.94582E-01
25	0.42113E-01	0.15022E-01	0.91577E-01
26	0.60263E-01	0.18150E-01	0.87947E-01
27	0.81417E-01	0.21154E-01	0.83717E-01
28	0.10543E+00	0.24013E-01	0.78914E-01
29	0.13214E+00	0.26708E-01	0.73572E-01
30	0.16136E+00	0.29221E-01	0.67728E-01
31	0.19289E+00	0.31534E-01	0.61421E-01
32	0.22653E+00	0.33632E-01	0.54695E-01
33	0.26203E+00	0.35500E-01	0.47595E-01
34	0.29915E+00	0.37126E-01	0.40170E-01
35	0.33765E+00	0.38498E-01	0.32470E-01
36	0.37726E+00	0.39607E-01	0.24549E-01

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37	0.41770E+00	0.40445E-01	0.16459E-01
38	0.45871E+00	0.41008E-01	0.82579E-02
39	0.50000E+00	0.41290E-01	-0.15705E-08
40	0.54129E+00	0.41290E-01	-0.82579E-02
41	0.58230E+00	0.41008E-01	-0.16459E-01
42	0.62274E+00	0.40445E-01	-0.24549E-01
43	0.66235E+00	0.39607E-01	-0.32470E-01
44	0.70085E+00	0.38498E-01	-0.40170E-01
45	0.73797E+00	0.37126E-01	-0.47595E-01
46	0.77347E+00	0.35500E-01	-0.54695E-01
47	0.80711E+00	0.33632E-01	-0.61421E-01
48	0.83864E+00	0.31534E-01	-0.67728E-01
49	0.86786E+00	0.29221E-01	-0.73572E-01
50	0.89457E+00	0.26708E-01	-0.78914E-01
51	0.91858E+00	0.24013E-01	-0.83717E-01
52	0.93974E+00	0.21154E-01	-0.87947E-01
53	0.95789E+00	0.18150E-01	-0.91577E-01
54	0.97291E+00	0.15022E-01	-0.94582E-01
55	0.98470E+00	0.11792E-01	-0.96940E-01
56	0.99318E+00	0.84805E-02	-0.98636E-01
57	0.99829E+00	0.51116E-02	-0.99658E-01
58	0.10000E+01	0.17078E-02	-0.10000E+00
59	0.10099E+01	0.98938E-02	0.00000E+00
60	0.10395E+01	0.29606E-01	0.00000E+00
61	0.10886E+01	0.49093E-01	0.00000E+00
62	0.11568E+01	0.68206E-01	0.00000E+00
63	0.12436E+01	0.86801E-01	0.00000E+00
64	0.13483E+01	0.10473E+00	0.00000E+00
65	0.14702E+01	0.12187E+00	0.00000E+00
66	0.16083E+01	0.13808E+00	0.00000E+00
67	0.17615E+01	0.15324E+00	0.00000E+00
68	0.19288E+01	0.16723E+00	0.00000E+00
69	0.21087E+01	0.17995E+00	0.00000E+00
70	0.23000E+01	0.19130E+00	0.00000E+00
71	0.25012E+01	0.20119E+00	0.00000E+00
72	0.27107E+01	0.20956E+00	0.00000E+00
73	0.29271E+01	0.21632E+00	0.00000E+00
74	0.31485E+01	0.22144E+00	0.00000E+00
75	0.33734E+01	0.22488E+00	0.00000E+00
76	0.36000E+01	0.22660E+00	0.00000E+00

Y GRID

j	Y(j)	Y(j)-Y(j-1)
1	-0.55668E-02	
2	0.00000E+00	0.55668E-02
3	0.55668E-02	0.55668E-02
4	0.22243E-01	0.16677E-01
5	0.49958E-01	0.27715E-01
6	0.88593E-01	0.38635E-01
7	0.13798E+00	0.49389E-01
8	0.19791E+00	0.59932E-01
9	0.26813E+00	0.70218E-01
10	0.34833E+00	0.80203E-01
11	0.43818E+00	0.89845E-01
12	0.53728E+00	0.99102E-01
13	0.64522E+00	0.10794E+00
14	0.76152E+00	0.11631E+00

```

15  0.88570E+00  0.12418E+00
16  0.10172E+01  0.13152E+00
17  0.11555E+01  0.13830E+00
18  0.13000E+01  0.14448E+00
19  0.14500E+01  0.15005E+00
20  0.16050E+01  0.15497E+00
21  0.17643E+01  0.15923E+00
22  0.19271E+01  0.16281E+00
23  0.20928E+01  0.16569E+00
24  0.22606E+01  0.16787E+00
25  0.24300E+01  0.16932E+00
26  0.26000E+01  0.17005E+00

```

Iterative Scheme Options:

```

1 - SOR
2 - SLOR
3 - AF-1
4 - AF-2

```

Select option:3

Iterative Scheme Selected: - AF-1

INPUT relaxation factor:

Suggested values for the ADI Relaxation Factors:

```

AF-1 - 2.0
AF-2 - 1.333          :2

```

INPUT number of iterations, maxit:60

Convergence History									
Residual					Correction				
ITER	IMAX	JMAX	MAX	AVRG	IMAX	JMAX	MAX	AVRG	
1	20	2	0.3593E+02	0.3711E+01	2	2	0.0000E+00	0.0000E+00	
2	22	2	0.1337E+02	0.1484E+01	22	2	0.3769E-03	0.3945E-04	
3	54	3	0.5960E+01	0.7183E+00	23	2	0.4242E-03	0.5228E-04	
4	53	2	0.4705E+01	0.5773E+00	24	3	0.7004E-03	0.1018E-03	
5	26	3	0.1822E+01	0.2557E+00	52	2	0.1256E-02	0.2001E-03	
6	50	2	0.1881E+01	0.2363E+00	27	3	0.1949E-02	0.3720E-03	
7	48	2	0.5231E+00	0.1102E+00	28	2	0.2974E-02	0.6784E-03	
8	50	2	0.6676E+00	0.9420E-01	29	3	0.3860E-02	0.1109E-02	
9	58	2	0.2509E+00	0.3698E-01	50	2	0.3971E-02	0.1458E-02	
10	65	2	0.1228E+00	0.1216E-01	17	4	0.3054E-02	0.1308E-02	
11	69	2	0.4270E-01	0.4370E-02	12	2	0.1963E-02	0.8412E-03	
12	2	2	0.2911E-01	0.2657E-02	7	4	0.1477E-02	0.7429E-03	
13	2	2	0.8172E-02	0.7845E-03	2	2	0.2889E-06	0.2579E-07	
14	2	3	0.2018E-02	0.3537E-03	2	2	0.1939E-06	0.2348E-07	
15	2	2	0.2251E-02	0.3438E-03	2	5	0.2281E-06	0.5201E-07	
16	2	8	0.9842E-03	0.2974E-03	2	8	0.5469E-06	0.1551E-06	
17	2	2	0.2147E-02	0.3181E-03	2	8	0.1539E-05	0.4865E-06	
18	2	3	0.9461E-03	0.2754E-03	2	26	0.4705E-05	0.1523E-05	
19	76	2	0.2082E-02	0.2908E-03	2	26	0.1504E-04	0.4714E-05	
20	2	3	0.8731E-03	0.2511E-03	2	26	0.4639E-04	0.1455E-04	

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21	2	2	0.1618E-02	0.2593E-03	2	26	0.1288E-03	0.4477E-04
22	2	3	0.5327E-03	0.1972E-03	2	26	0.3090E-03	0.1293E-03
23	2	2	0.7191E-03	0.1381E-03	2	26	0.6491E-03	0.3123E-03
24	2	2	0.3744E-03	0.1247E-03	76	2	0.8472E-08	0.2005E-08
25	76	3	0.2757E-03	0.1209E-03	76	2	0.1530E-07	0.6149E-08
26	2	2	0.2809E-03	0.1218E-03	76	3	0.4273E-07	0.1975E-07
27	2	3	0.2541E-03	0.1202E-03	2	2	0.1338E-06	0.6388E-07
28	2	2	0.2690E-03	0.1209E-03	2	3	0.4245E-06	0.2065E-06
29	2	3	0.2494E-03	0.1193E-03	2	2	0.1365E-05	0.6678E-06
30	2	2	0.2632E-03	0.1194E-03	2	3	0.4378E-05	0.2152E-05
31	76	3	0.2396E-03	0.1156E-03	2	2	0.1392E-04	0.6873E-05
32	76	2	0.2413E-03	0.1102E-03	2	3	0.4324E-04	0.2143E-04
33	2	3	0.1930E-03	0.9282E-04	2	2	0.1262E-03	0.6244E-04
34	2	2	0.1223E-03	0.5565E-04	2	4	0.3006E-03	0.1480E-03
35	76	2	0.1144E-03	0.5498E-04	76	2	0.1834E-08	0.8544E-09
36	2	3	0.1122E-03	0.5505E-04	76	2	0.5671E-08	0.2763E-08
37	2	2	0.1125E-03	0.5494E-04	2	3	0.1826E-07	0.8962E-08
38	2	3	0.1120E-03	0.5503E-04	2	24	0.5920E-07	0.2907E-07
39	2	2	0.1124E-03	0.5491E-04	2	24	0.1920E-06	0.9428E-07
40	2	3	0.1118E-03	0.5493E-04	2	25	0.6223E-06	0.3056E-06
41	2	2	0.1117E-03	0.5459E-04	2	26	0.2013E-05	0.9883E-06
42	2	3	0.1096E-03	0.5387E-04	2	26	0.6467E-05	0.3175E-05
43	76	2	0.1047E-03	0.5121E-04	2	26	0.2032E-04	0.9979E-05
44	2	3	0.8895E-04	0.4380E-04	2	26	0.5953E-04	0.2927E-04
45	2	2	0.5280E-04	0.2587E-04	2	26	0.1414E-03	0.6957E-04
46	2	2	0.5261E-04	0.2589E-04	2	2	0.8166E-09	0.4005E-09
47	2	3	0.5256E-04	0.2584E-04	2	2	0.2643E-08	0.1300E-08
48	2	2	0.5256E-04	0.2588E-04	2	3	0.8569E-08	0.4216E-08
49	2	3	0.5254E-04	0.2584E-04	2	2	0.2779E-07	0.1367E-07
50	76	2	0.5253E-04	0.2586E-04	2	3	0.9014E-07	0.4435E-07
51	2	3	0.5244E-04	0.2579E-04	2	2	0.2921E-06	0.1437E-06
52	2	2	0.5221E-04	0.2570E-04	2	3	0.9448E-06	0.4649E-06
53	2	3	0.5142E-04	0.2529E-04	2	2	0.3035E-05	0.1493E-05
54	2	2	0.4897E-04	0.2411E-04	2	3	0.9538E-05	0.4693E-05
55	76	3	0.4178E-04	0.2055E-04	2	2	0.2797E-04	0.1376E-04
56	2	2	0.2471E-04	0.1218E-04	2	3	0.6648E-04	0.3271E-04
57	2	2	0.2471E-04	0.1215E-04	2	2	0.3829E-09	0.1884E-09
58	76	3	0.2471E-04	0.1216E-04	76	2	0.1242E-08	0.6109E-09
59	76	2	0.2470E-04	0.1215E-04	76	3	0.4028E-08	0.1982E-08
60	76	3	0.2470E-04	0.1216E-04	2	25	0.1307E-07	0.6429E-08

PRESSURE DISTRIBUTION ON THE AIRFOIL

I	x/c	U	Cp
21	0.00085	-0.10158	0.20316
22	0.00426	-0.08440	0.16880
23	0.01106	-0.06486	0.12972
24	0.02120	-0.04688	0.09376
25	0.03460	-0.03086	0.06171
26	0.05119	-0.01676	0.03352
27	0.07084	-0.00431	0.00861
28	0.09342	0.00679	-0.01357
29	0.11878	0.01670	-0.03340
30	0.14675	0.02556	-0.05111
31	0.17713	0.03344	-0.06687
32	0.20971	0.04039	-0.08078
33	0.24428	0.04644	-0.09289

34	0.28059	0.05162	-0.10324
35	0.31840	0.05593	-0.11186
36	0.35745	0.05938	-0.11876
37	0.39748	0.06196	-0.12393
38	0.43821	0.06369	-0.12738
39	0.47936	0.06455	-0.12910
40	0.52064	0.06455	-0.12910
41	0.56179	0.06369	-0.12738
42	0.60252	0.06196	-0.12393
43	0.64255	0.05938	-0.11876
44	0.68160	0.05593	-0.11186
45	0.71941	0.05162	-0.10324
46	0.75572	0.04644	-0.09289
47	0.79029	0.04039	-0.08078
48	0.82287	0.03344	-0.06687
49	0.85325	0.02556	-0.05111
50	0.88122	0.01670	-0.03340
51	0.90658	0.00679	-0.01357
52	0.92916	-0.00431	0.00861
53	0.94881	-0.01676	0.03352
54	0.96540	-0.03086	0.06171
55	0.97880	-0.04688	0.09376
56	0.98894	-0.06486	0.12972
57	0.99574	-0.08440	0.16880
58	0.99915	-0.10158	0.20316

Additional iteration? (Y/N):n

STOP

Details of THINFOIL procedures: by Valery Razgonyaev

Basic notation: The standard or delta form of the iterative scheme is:

$$NC_{i,j}^n + \omega L\phi_{i,j}^n = 0,$$

where

$$C_{i,j}^n = \phi_{i,j}^{n+1} - \phi_{i,j}^n$$

Finite difference representation of the Laplace's operator and residual is given by:

$$L\phi_{i,j} = \delta_{xx}\phi_{i,j}^n + \delta_{yy}\phi_{i,j}^n = 0$$

where

$$\delta_{xx}\phi_{i,j}^n = Ax_{i,j}\phi_{i-1,j}^n + Bx_{i,j}\phi_{i,j}^n + Cx_{i,j}\phi_{i+1,j}^n$$

$$\delta_{yy}\phi_{i,j}^n = Ay_{i,j}\phi_{i,j-1}^n + By_{i,j}\phi_{i,j}^n + Cy_{i,j}\phi_{i,j+1}^n$$

$$Ax_{i,j} = \frac{2}{\left[(x_{i+1,j} - x_{i-1,j})(x_{i,j} - x_{i-1,j}) \right]}$$

$$Cx_{i,j} = \frac{2}{\left[(x_{i+1,j} - x_{i-1,j})(x_{i+1,j} - x_{i,j}) \right]}$$

$$Bx_{i,j} = -(Ax_{i,j} + Cx_{i,j})$$

and $Ay_{i,j}$, $By_{i,j}$, $Cy_{i,j}$ are defined in similar manner. Other definitions:

$$E^{x\pm 1}\phi_{i,j}^n = \phi_{i\pm 1,j}^n$$

$$E^{y\pm 1}\phi_{i,j}^n = \phi_{i,j\pm 1}^n$$

$$\delta_x^+\phi_{i,j}^n = \frac{1}{(x_{i+1,j} - x_{i,j})} E^{x+1}\phi_{i,j}^n$$

$$\delta_x^-\phi_{i,j}^n = \frac{1}{(x_{i,j} - x_{i-1,j})} E^{x-1}\phi_{i,j}^n$$

and δ_y^\pm is defined in similar manner.

$NX+1$, $NY+1$ - numbers of points in x and y directions respectively.

N_{LE} , N_{TE} - values of the i index corresponding to LE and TE respectively.

Boundary Conditions: The boundary condition

$$\frac{\partial \phi}{\partial n} = \frac{df}{dx},$$

where

$$f(x) = \frac{t}{c} \frac{x}{c} \left(1 - \frac{x}{c} \right)$$

is applied on the line $y = 0$ for $0 < x/c < 1$, while on all other boundaries:

$$\frac{\partial \phi}{\partial n} = 0.$$

The implementation of the boundary condition using so-called “dummy rows“ is discussed in the text, and in our case we define:

$$\phi_{i,1}^n = \phi_{i,3}^n - (y_{i,3} - y_{i,1}) \left(\frac{t}{c} \right) (1 - 2x_{i,2}) \quad \text{for } N_{LE} < i < N_{TE}$$

$$\phi_{i,1}^n = \phi_{3,j}^n \quad \text{for } 2 < i < N_{LE} \text{ and } N_{TE} < i < NX$$

$$\phi_{i,NY+1}^n = \phi_{i,NY-1}^n \quad \text{for } 2 < i < NX$$

$$\phi_{1,j}^n = \phi_{3,j}^n \quad \text{and} \quad \phi_{NX+1,j}^n = \phi_{NX-1,j}^n \quad \text{for } 2 < j < NY$$

Since we use the standard or delta form for the iterative scheme expressions we need to have our B.C. expressed in terms of $C_{i,j}^n$. The simplest way to accomplish this is to define:

$$\phi_{i,1}^1 = -(y_{i,3} - y_{i,1}) \left(\frac{t}{c} \right) (1 - 2x_{i,2}) \quad \text{for } N_{LE} < i < N_{TE}$$

$$\phi_{i,j}^1 = 0 \quad \text{elsewhere}$$

Then B.C. for $C_{i,j}^n$ can be written as:

$$C_{i,1}^n = C_{i,3}^n \quad \text{and} \quad C_{i,NY+1}^n = C_{i,NY-1}^n \quad \text{for } 2 < i < NX$$

$$C_{1,j}^n = C_{3,j}^n \quad \text{and} \quad C_{NX+1,j}^n = C_{NX-1,j}^n \quad \text{for } 2 < j < NY$$

Note that $C_{i,1}^n = \phi_{i,1}^{n+1} - \phi_{i,1}^n =$

$$\begin{aligned} & \left(\phi_{i,1}^{n+1} - (y_{i,3} - y_{i,1}) \left(\frac{t}{c} \right) (1 - 2x_{i,2}) \right) - \left(\phi_{i,3}^n - (y_{i,3} - y_{i,1}) \left(\frac{t}{c} \right) (1 - 2x_{i,2}) \right) = \\ & = \phi_{i,3}^{n+1} - \phi_{i,3}^n = C_{i,3}^n \end{aligned}$$

Point Gauss-Seidel Iterative Scheme (SOR): Scheme definition:

$$\begin{aligned} \hat{\phi}_{i,j}^{n+1} &= -\left(Ax_{i,j} \phi_{i-1,j}^{n+1} + Cx_{i,j} \phi_{i+1,j}^n + Ay_{i,j} \phi_{i,j-1}^{n+1} \right) / \left(Bx_{i,j} + By_{i,j} \right) \\ \phi_{i,j}^{n+1} &= \phi_{i,j}^n + \omega \left(\hat{\phi}_{i,j}^{n+1} - \phi_{i,j}^n \right) \end{aligned}$$

or

$$\left(Bx_{i,j} + By_{i,j} \right) C_{i,j}^n = -\omega \left(L\phi_{i,j} + Ax_{i,j} C_{i-1,j}^n + Ay_{i,j} C_{i,j-1}^n \right)$$

Values $C_{i-1,j}^n$ and $C_{i,j-1}^n$ are already known at the time $C_{i,j}^n$ is computed. Note, that at the beginning of each iteration, when $i = j = 2$, values $C_{1,2}^n$ and $C_{2,1}^n$ are undefined yet and, thus, should be set to 0.

Operator N is defined by:

$$N = \left(Bx_{i,j} + By_{i,j} \right) I + \omega \left(Ax_{i,j} E^{x-1} + Ay_{i,j} E^{y-1} \right)$$

Line Gauss-Seidel Iterative Scheme (SLOR): Scheme definition:

$$\hat{\phi}_{i,j}^{n+1} = -(A_{x_{i,j}} \phi_{i-1,j}^{n+1} + C_{x_{i,j}} \phi_{i+1,j}^n + A_{y_{i,j}} \hat{\phi}_{i,j-1}^{n+1} + C_{y_{i,j}} \hat{\phi}_{i,j+1}^{n+1}) / (B_{x_{i,j}} + B_{y_{i,j}})$$

$$\phi_{i,j}^{n+1} = \phi_{i,j}^n + \omega (\hat{\phi}_{i,j}^{n+1} - \phi_{i,j}^n)$$

or

$$A_{y_{i,j}} C_{i,j-1}^n + (B_{x_{i,j}} + B_{y_{i,j}}) C_{i,j}^n + C_{y_{i,j}} C_{i,j+1}^n = -\omega (L\phi_{i,j} + A_{x_{i,j}} C_{i-1,j}^n)$$

Value $C_{i-1,j}^n$ is already known at the time $C_{i,j}^n$ is computed and, thus, update in i -th column is found by inverting tridiagonal matrix. At the beginning of each iteration, when $i=2$, value $C_{1,j}^n$ is undefined yet and, thus, should be set to 0, but at the end, when $i=NX$, from the B.C.

$C_{NX+1,j}^n = C_{NX-1,j}^n$. Exploitation of the B.C. for $j=2$ and $j=NY$ defines $C_{i,1}^n = C_{i,3}^n$ and

$C_{i,NY+1}^n = C_{i,NY-1}^n$.

Operator N is defined by:

$$N = (B_{x_{i,j}} + B_{y_{i,j}})I + A_{y_{i,j}} E^{y-1} + C_{y_{i,j}} E^{y+1} + \omega A_{x_{i,j}} E^{x-1}$$

Approximate Factorisation AF-1 Iterative Scheme (ADI): Scheme definition:

$$(\delta_{xx} - \alpha)(\delta_{yy} - \alpha)C_{i,j}^n = \omega \alpha L\phi_{i,j}$$

where α is a parameter.

Operator N is defined by:

$$N = L - \alpha I - \frac{1}{\alpha} \delta_{xx} \delta_{yy}$$

The required equations are solved in two steps:

1) Inversion of the outer operator $N_{out} = \delta_{xx} - \alpha I$ or sweep along i direction by letting $F_{i,j}^n = (\delta_{yy} - \alpha)C_{i,j}^n$ and solving directly for $F_{i,j}^n$ the tridiagonal matrix equation:

$$A_{x_{i,j}} F_{i-1,j}^n + (B_{x_{i,j}} - \alpha) F_{i,j}^n + C_{x_{i,j}} F_{i+1,j}^n = \omega \alpha L\phi_{i,j}$$

It can be shown that B.C. for function $F_{i,j}^n$ are the same as for $C_{i,j}^n$, thus, $F_{1,j}^n = F_{3,j}^n$ and

$F_{NX+1,j}^n = F_{NX-1,j}^n$.

2) Inversion of the inner operator $N_{in} = \delta_{yy} - \alpha I$ or sweep along j direction by solving directly for

$C_{i,j}^n$ the tridiagonal matrix equation:

$$Ay_{i,j} * C_{i,j-1}^n + (By_{i,j} - \alpha) * C_{i,j}^n + Cy_{i,j} * C_{i,j+1}^n = F_{i,j}^n$$

with B.C. $C_{i,1}^n = C_{i,3}^n$ and $C_{i,NY+1}^n = C_{i,NY-1}^n$.

For uniform Cartesian grid and periodic B.C. the optimal value of ω is 2. To speed up convergence Ballhaus suggests to use a sequence of parameters $\alpha_k = \alpha_1 (\alpha_h)^{(M-k)/(M-1)}$, $\alpha_1 = 1$, $\alpha_h = 4/\Delta y^2$ $k=1, \dots, M$ instead of one parameter α . In the code $M=11$ and $\Delta y = Y(3) - Y(2)$.

Approximate Factorisation AF-2 Iterative Scheme (ADI): Scheme definition:

$$(\alpha - \delta_x^+) (\delta_{yy} - \alpha \delta_x^-) C_{i,j}^n = -\omega \alpha L \phi_{i,j}$$

where α is a parameter.

Operator N is defined by:

$$N = L - \alpha \delta_x^+ - \frac{1}{\alpha} \delta_x^- \delta_{yy}$$

The required equations are solved in two steps:

1) Inversion of the outer operator $N_{out} = \alpha I - \delta_x^+$ or sweep along i direction by letting $F_{i,j}^n = (\delta_{yy} - \alpha \delta_x^-) C_{i,j}^n$ and solving directly for $F_{i,j}^n$ the bidiagonal matrix equation:

$$(\alpha + 1/(x_{i+1,j} - x_{i,j})) * F_{i,j}^n - 1/(x_{i+1,j} - x_{i,j}) * F_{i+1,j}^n = -\omega \alpha L \phi_{i,j}$$

It can be shown that B.C. for function $F_{i,j}^n$ are the same as for $C_{i,j}^n$, thus, $F_{NX+1,j}^n = F_{NX-1,j}^n$.

2) Inversion of the inner operator $N_{in} = \delta_{yy} - \alpha \delta_x^-$ or sweep along j direction by solving directly for $C_{i,j}^n$ the tridiagonal matrix equation:

$$Ay_{i,j} * C_{i,j-1}^n + (By_{i,j} - \alpha/(x_{i,j} - x_{i-1,j})) * C_{i,j}^n + Cy_{i,j} * C_{i,j+1}^n = F_{i,j}^n - \alpha/(x_{i,j} - x_{i-1,j}) * C_{i-1,j}^n$$

with B.C. $C_{i,1}^n = C_{i,3}^n$, $C_{i,NY+1}^n = C_{i,NY-1}^n$.

Value $C_{i-1,j}^n$ becomes available before $C_{i,j}^n$ is computed, except for the case $i=2$, when it should be set to 0.

Suggested value of ω is 4/3. Similarly to AF-1 scheme a sequence of parameters

$\alpha_k = \alpha_1 (\alpha_h)^{(M-k)/(M-1)}$, $\alpha_1 = 1$, $\alpha_h = 2/\Delta y$ $k=1, \dots, M$ recommended by Ballhaus is used with $M=11$ and $\Delta y = Y(3) - Y(2)$.