2. Getting Ready for Computational Aerodynamics: Fluid Mechanics Foundations

from AIAA 82-0315, by D.R. Carlson

We need to review the governing equations of fluid mechanics before examining the methods of computational aerodynamics in detail. Developments in computational methods have resulted in a slightly different approach to the fundamental conservation statements compared with pre-computer classical presentations. The review also establishes the nomenclature to be used in the rest of the chapters. The presentation presumes that the reader has previously had a course in fluid mechanics or aerodynamics. Many excellent discussions of the foundations of fluid mechanics for aerodynamics application are available. Karamcheti[1] does a good job. Other books containing good discussions of the material include the books by Bertin and Smith,[2] Anderson,[3] and Moran.[4] The best formal derivation of the equations is by Grossman.[5]

2.1 Governing Equations of Fluid Mechanics

The flow is assumed to be a continuum. For virtually all aerodynamics work this is a valid assumption. One case where this may not be true: rarefied gas dynamics, where the flow has such low density that the actual molecular motion must be analyzed. This is rarely important, even in aero-space plane calculations. Aeroassisted Orbital Transfer Vehicles (AOTV's) are the only current vehicles requiring non-continuum flowfield analysis.

The fluid is defined by an equation of state and the thermodynamic and transport properties, *i.e.*, the ratio of specific heats, γ, viscosity, μ, and the coefficient of heat conduction, k. Governing equations and boundary conditions control the motion of the fluid. The governing equations are given by conservation laws:

• mass	continuity
• momentum	Newton's 2^{nd} Law, $\mathbf{F}=m\mathbf{a}$
• energy	1st Law of Thermodynamics

Coordinate systems are also important in aerodynamics. The general equations of fluid motion are independent of the coordinate system. However, simplifying assumptions frequently introduce a *directional bias* into approximate forms of the equations, and require that they be used with a specific coordinate system orientation relative to the flowfield.

Cartesian coordinates are normally used to describe vehicle geometry. In this chapter we will work entirely in the Cartesian coordinate system. It is frequently desirable to make calculations in non-Cartesian coordinate systems that are distorted to fit a particular shape. General non-orthogonal curvilinear coordinates are discussed in Chapter 9. Even when using Cartesian coordinates, the x, y, and z coordinates are oriented differently depending on whether the flow is two- or three-dimensional. Figure 2-1 shows the usual two-dimensional coordinate system. The standard aerodynamics coordinate system in three dimensions is illustrated in Fig. 2-2.

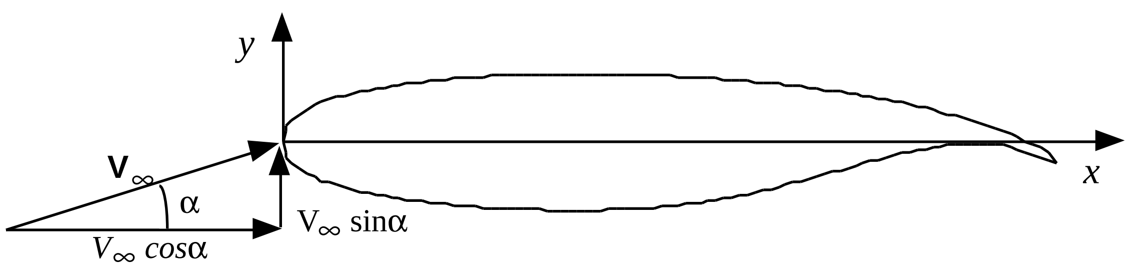

Figure 2-1 Coordinate system for two-dimensional flow.

In general Cartesian coordinates, the independent variables are x, y, z, and t. We want to know the velocities, u, v, w, and the fluid properties; p, ρ, T. These six unknowns require six equations. The six equations used are provided by the following:

continuity	1	equation(s)
momentum	3	"
energy	1	"
equation of state	1	" .

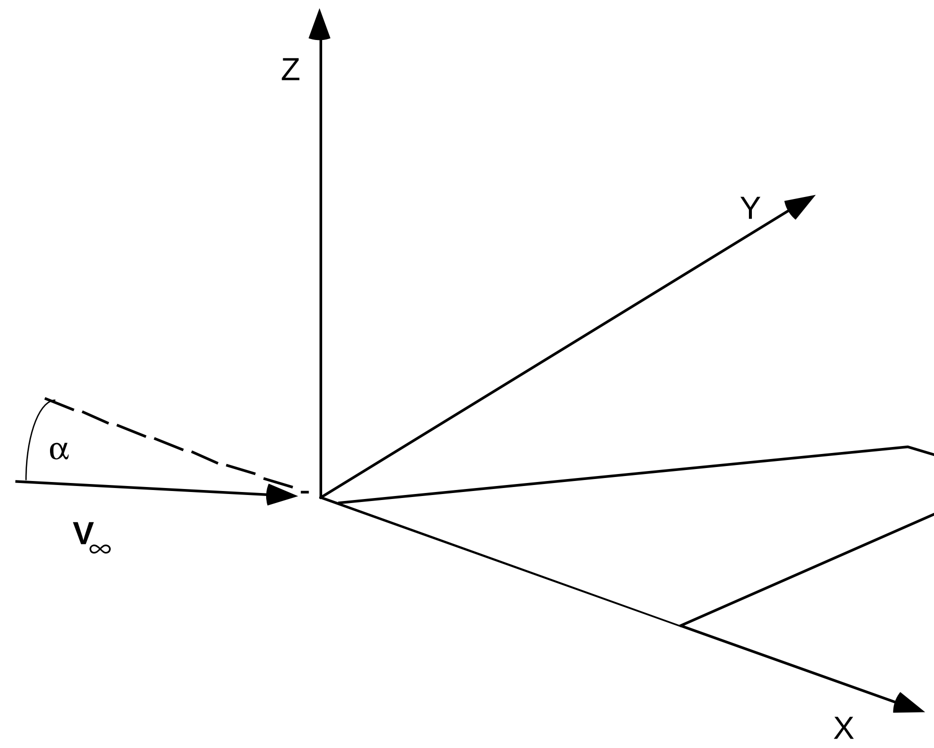

Figure 2-2 Standard coordinate system for three-dimensional flow.

Assumptions frequently reduce the number of equations required. Examples include incompressible, inviscid, irrotational flow, which can be described by a single equation, as shown below. Prior to the 1980s almost all aerodynamics work used a single partial differential equation, possibly coupled with another equation. An example of this approach is the calculation of potential flow for the inviscid portion of the flowfield, and use of the boundary layer equations to compute the flowfield where an estimate of the viscous effects is required.

2.2 Derivation of Governing Equations

We now need to develop a mathematical model of the fluid motion suitable for use in numerical calculations. We want to find the flowfield velocity, pressure and temperature distributions. The mathematical model is based on the conservation laws and the fluid properties, as stated above. Two approaches can be used to obtain the mathematical description defining the governing equations.

I. *Lagrangian*: In this method each fluid particle is traced as it moves around the body. Even in steady flow, the forces encountered by the particle will be a function of its time history as it moves relative to a coordinate system fixed to the body, as defined in Figs. 2-1 and 2-2. This method corresponds to the conventional concept of Newton's Second Law.

II. *Eulerian*: In this method we look at the entire space around the body as a field, and determine flow properties at various points in the field while the fluid particles stream past. Once this viewpoint is adopted, we consider the distribution of velocity and pressure throughout the field, and ignore the motion of individual fluid particles.

Virtually all computational aerodynamics methods use the Eulerian approach. The use of this approach requires careful attention in the application of the conservation concepts, and Newton's second law in particular. Since these two approaches describe the same physical phenomena, they can be mathematically related. Karamcheti[1] provides a particularly good explanation of the ideas underlying approaches to the governing equations in his Chapters 4-7. Newton's Law governs the motion of a fixed fluid particle. However, to establish a viable method for computation, aerodynamicists employ the Eulerian approach, and define a control volume, which maintains a fixed location relative to the coordinate system. The connection between the rate of change of the properties of the fixed fluid particle (velocity, density, pressure, *etc*.) and the rate of change of fluid properties flowing through a fixed control volume[*] requires special consideration. The substantial derivative, discussed below, is employed to define the rate of change of fixed fluid particle properties as the particle moves through the flowfield relative to the fixed coordinate system. An integral approach to the description of the change of properties of a fluid particle relative to the fixed coordinate system is available through the use of the Reynolds Transport Theorem, which is described by Owczarek[6] and Grossman[5] (section 1.2).

The conservation equations can be expressed in either a differential or integral viewpoint. The differential form is the most frequently used in fluid mechanics analysis and textbooks. However, many numerical methods use the integral form. Numerically, integrals are more accurately computed than derivatives. The integral form handles discontinuities (shocks) better. The differential form assumes properties are continuous. We will use aspects of each approach.

* The concept of a "control volume" arose as an engineering requirement for a means to formulate the physical description to allow calculations to be made. It differs from the viewpoint adopted by physicists. An explanation of the concept's origins is contained in the book by Walter G. Vincenti, *What Engineers Know, and How They Know It: Analytical Studies from Aeronautical History*, John Hopkins Univ. Press, 1990. The chapter is entitled "A Theoretical Tool for Design: Control Volume Analysis 1912-1953."

2.2.1 Conservation of Mass: the Continuity Equation

In this section we derive the continuity equation from a control volume viewpoint (in 2D), and then we look at the equivalent integral statement and the use of the Gauss Divergence Theorem to establish the connection. Other derivations are given by Moran[4] (sections 2.2, 2.3, 2.4) Anderson[3] (chapters 2 and 6), and Bertin and Smith[2] (chapter 2).

The statement of conservation of mass is in words simply:

net outflow of mass		*decrease of mass*
through the surface	=	*within the*
surrounding the volume		*volume.*

To translate this statement into a mathematical form, consider the control volume given in Fig. 2-3. Here, u is the velocity in the x-direction, v is the velocity in the y-direction, and ρ is the density.

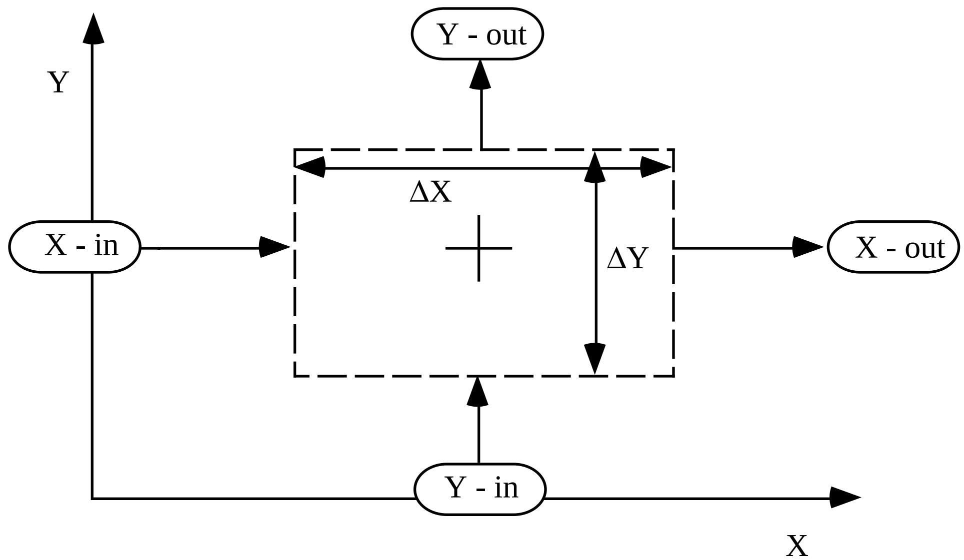

Figure 2-3. Control volume for conservation of mass.*

The net mass flow rate, or flux,** (*out* of the volume) is:

$$[\text{X - out}] - [\text{X - in}] + [\text{Y - out}] - [\text{Y - in}] = \text{change of mass (decrease)}$$

$$= -\frac{\partial \rho}{\partial t}\Delta X \Delta Y. \qquad (2\text{-}1)$$

* Note that convention requires that control volumes be described using dashed lines to illustrate that the boundaries are fictitious, and fluid is flowing freely across them.

** A *flux* is a quantity which flows across the boundary of a defined surface. Typically we think of mass, momentum and energy fluxes.

Use a Taylor series expansion of the mass fluxes into the volume around the origin of the volume. The flux per unit length through the surface is multiplied by the length of the surface to get:

$$[\text{X - out}] = \left[\rho u + \frac{\partial \rho u}{\partial x} \cdot \frac{\Delta X}{2}\right]\Delta Y$$

$$[\text{X - in}] = \left[\rho u - \frac{\partial \rho u}{\partial x} \cdot \frac{\Delta X}{2}\right]\Delta Y$$

$$[\text{Y - out}] = \left[\rho v + \frac{\partial \rho v}{\partial y} \cdot \frac{\Delta Y}{2}\right]\Delta X$$

$$[\text{Y - in}] = \left[\rho v - \frac{\partial \rho v}{\partial y} \cdot \frac{\Delta Y}{2}\right]\Delta X \ . \tag{2-2}$$

Adding these terms up we get:

$$\left[\rho u + \frac{\partial \rho u}{\partial x} \cdot \frac{\Delta X}{2}\right]\Delta Y - \left[\rho u - \frac{\partial \rho u}{\partial x} \cdot \frac{\Delta X}{2}\right]\Delta Y$$

$$+ \left[\rho v + \frac{\partial \rho v}{\partial y} \cdot \frac{\Delta Y}{2}\right]\Delta X - \left[\rho v - \frac{\partial \rho v}{\partial y} \cdot \frac{\Delta Y}{2}\right]\Delta X = -\frac{\partial \rho}{\partial t}\Delta X \Delta Y \, . \tag{2-3}$$

Summing up and canceling $\Delta X \Delta Y$ we get:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = -\frac{\partial \rho}{\partial t} \tag{2-4}$$

or in three dimensions:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \, . \tag{2-5}$$

This is the differential form of the continuity equation. The more general vector form of the equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \, . \qquad \text{(2-6}$$

Alternately, consider the arbitrary control volume shown in Fig. 2-4. The conservation of mass can then be written in an integral form quite simply. The surface integral of the flow out of the volume simply equals the change of mass given in the volume:

$$\oiint \rho \mathbf{V} \cdot \hat{n}\, dS = -\frac{\partial}{\partial t} \iiint_{v} \rho \, dV \, . \tag{2-7}$$

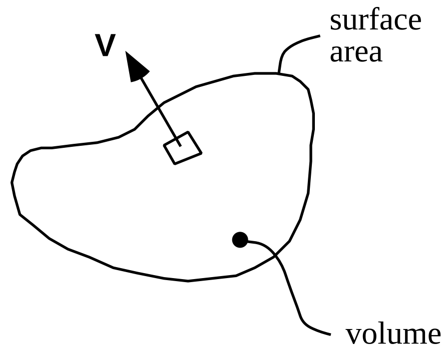

Figure 2-4. Arbitrary fluid control volume.

This is true without making any assumption requiring continuous variables and differentiability. It's for all flows, viscous or inviscid, compressible or incompressible.

To relate this expression to the differential form, we make use of the Gauss Divergence Theorem, which assumes continuous partial derivatives. It is given by:

$$\oiint \mathbf{A} \cdot \hat{n}\, dS = \iiint_v \nabla \cdot \mathbf{A}\, dV \tag{2-8}$$

and the equivalent statement for a scalar is:

$$\oiint \phi\, \mathbf{n}\, dS = \iiint_v \operatorname{grad}\phi\, dV\,. \tag{2-9}$$

Using this theorem, the differential and integral forms can be shown to be the same. First, rewrite the surface integral in the conservation of mass, Eq. (2-7), as:

$$\oiint \rho \mathbf{V} \cdot \hat{n}\, dS = \iiint_v \nabla \cdot (\rho \mathbf{V})\, dV \tag{2-10}$$

using the divergence theorem, Eq. (2-8). The continuity equation integral form thus becomes:

$$\iiint_v \nabla \cdot (\rho \mathbf{V})\, dV = -\frac{\partial}{\partial t} \iiint_v \rho\, dV \tag{2-11}$$

and since v refers to a fixed volume, we can move $\partial/\partial t$ inside the integral,

$$\iiint_v \left[\nabla \cdot (\rho \mathbf{V}) + \frac{\partial \rho}{\partial t} \right] dV = 0\,. \tag{2-12}$$

For this to be true in general, the integrand must be zero, which is just the differential form! Further discussion, and other derivations are available in Moran,[4] sections 2.2, 2.3, and 2.4, Anderson,[3] section 2.6, and Bertin and Smith[2], Chapter 2.

2.2.2 Conservation of Momentum, and the Substantial Derivative

In this section we derive the general equations for the conservation of momentum. This is a statement of Newton's 2^{nd} Law: *The time rate of change of momentum of a body equals the net force exerted on it.* For a fixed mass this is the famous equation

$$\mathbf{F} = m\mathbf{a} = m\frac{D\mathbf{V}}{Dt}. \tag{2-13}$$

Substantial Derivative

We need to apply Newton's Law to a moving fluid element (the "body" in the 2^{nd} Law statement given above) from our fixed coordinate system. This introduces some extra complications. From our fixed coordinate system, look at what D/Dt means. Consider Fig. 2-5 (from Karamcheti[1]). Consider any fluid property, Q($\mathbf{r}$,t).

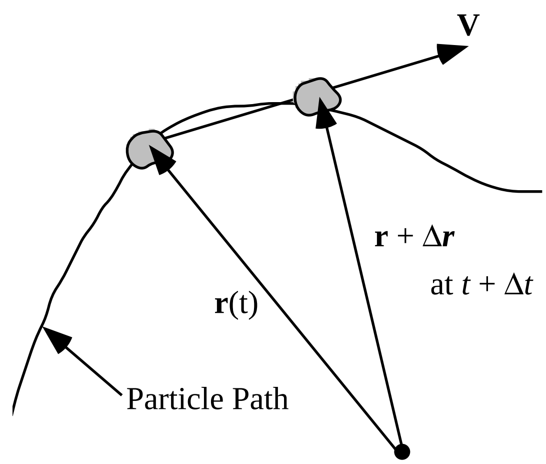

Figure 2-5. Moving particle viewed from a fixed coordinate system.

The change in position of the particle between the position r at t, and r+Δr at t+Δt is:

$$\Delta Q = Q(\mathbf{r} + \Delta s, t + \Delta t) - Q(\mathbf{r}, t). \tag{2-14}$$

The space change Δs is simply equal to $\mathbf{V}\Delta t$. Thus we can write:

$$\Delta Q = Q(\mathbf{r} + \mathbf{V}\Delta t, t + \Delta t) - Q(\mathbf{r}, t), \tag{2-15}$$

which is in a form which can be used to find the rate of change of Q:

$$\frac{DQ}{Dt} = \lim_{\Delta t \to 0}\frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0}\frac{Q(\mathbf{r} + \mathbf{V}\Delta t, t + \Delta t) - Q(\mathbf{r}, t)}{\Delta t}. \tag{2-16}$$

Note that the rate of change is in two parts, one for a change in time, and one for a change in space. Thus we write the change of Q as a function of both time and space using the Taylor series expansion as:

$$Q(\mathbf{r}+\mathbf{V}\Delta t, t+\Delta t) = Q(\mathbf{r},t) + \left.\frac{\partial Q}{\partial t}\right|_{\mathbf{r},t}\Delta t + \ldots + \left.\frac{\partial Q}{\partial s}\right|_{\mathbf{r},t} V\Delta t + \ldots, \tag{2-17}$$

where the direction of *s* is understood from Fig. 2-5. Substituting into Eq. (2-16) and taking the limit, we obtain:

$$\lim_{\Delta t \to 0}\frac{\Delta Q}{\Delta t} = \underbrace{\underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{local time}\\ \text{derivative, or}\\ \textit{local derivative}}} + \underbrace{\frac{\partial Q}{\partial s}V}_{\substack{\text{variation with}\\ \text{change of position,}\\ \textit{convective derivative}}}}_{\text{substantial derivative}}. \tag{2-18}$$

This is the important consideration in applying Newton's Law for a moving particle to a point fixed in a stationary coordinate system. The second term in Eq. (2-18) has the unknown velocity *V* multiplying a term containing the unknown *Q*. This is important.

The convective derivative introduces a fundamental nonlinearity into the system

We now put this result into a specific coordinate system:

$$\frac{\partial Q}{\partial s} = \mathbf{e}_V \cdot \nabla Q. \tag{2-19}$$

where $\mathbf{e}_V$ denotes the unit vector in the direction of **V**. Thus, $\mathbf{V} = V\mathbf{e}_V$ and:

$$\frac{\partial Q}{\partial s}V = \mathbf{V}\cdot\nabla Q. \tag{2-20}$$

Thus, we write the substantial derivative, Eq. (2-16), using Eqs.(2-18) and (2-20) as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla), \tag{2-21}$$

which can be applied to either a scalar as:

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\mathbf{V}\cdot\nabla)Q \tag{2-22}$$

or to a vector quantity as:

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}. \tag{2-23}$$

In Cartesian coordinates, $\mathbf{V} = u, v, w$, and the substantial derivative becomes:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}. \tag{2-24}$$

To solve equations containing these nonlinear terms we generally have to either use finesse, where we avoid solutions requiring Eq. (2-24) by using other facts about the flowfield to avoid having to deal with Eq.(2-24) directly, or employ numerical methods. There are only a very few special cases where you can obtain analytic solutions to equations explicitly including the non-linearity.

Forces

Now we need to find the net forces on the system. What are they?

- body forces
- pressure forces
- shear forces

Each of these forces applies to the control volume shown in Fig. 2-6 given below. The τ is a general symbol for stresses. In the figure, the first subscript indicates the direction normal to the surface, and the second subscript defines the direction in which the force acts. Fluids of interest in aerodynamics are *isotropic.* To satisfy equilibrium of moments about each axis:

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{zx} = \tau_{xz}. \tag{2-25}$$

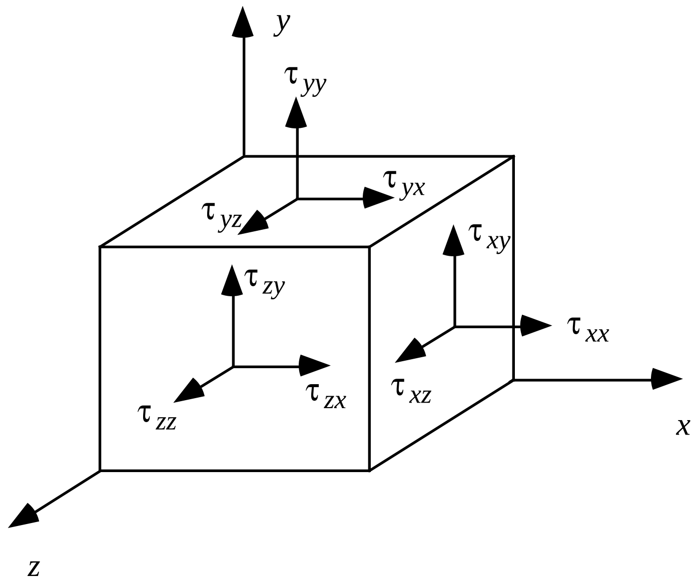

Figure 2-6. Control volume with surface forces shown.

The connection between pressure and stress is defined more specifically when the properties of a fluid are prescribed. Figure 2-7 shows the details of the forces, expanded about the origin using a Taylor Series. The force *f* is defined to be the body force per unit mass.

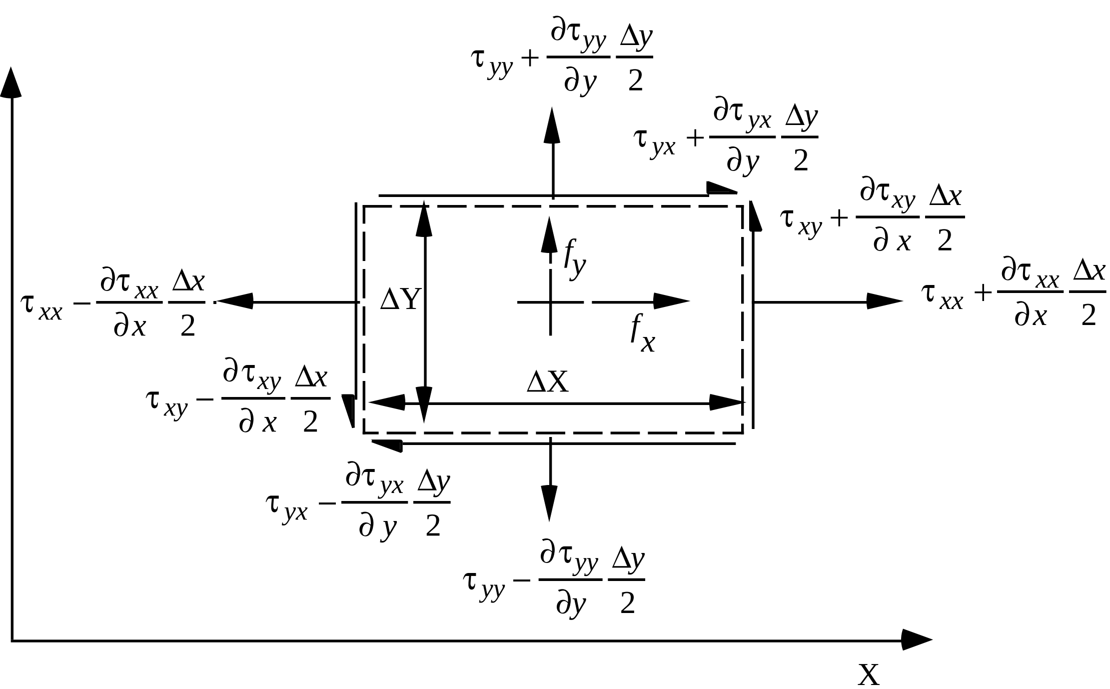

Figure 2-7. Details of forces acting on a two-dimensional control volume.

Considering the x-direction as an example, and using the Taylor series expansion shown in Figure 2-7, the net forces are found in a manner exactly analogous to the approach used in the derivation of the continuity equation. Thus, the net force in the x-direction is found to be:

$$\rho \cdot \Delta x \, \Delta y \, f_x + \frac{\partial}{\partial x}(\tau_{xx}) \Delta x \, \Delta y + \frac{\partial}{\partial y}(\tau_{yx}) \Delta y \, \Delta x . \qquad (2\text{-}26)$$

Now we combine the forces, including the z-direction terms. Substitute for the forces into the original statement, of $\mathbf{F} = m\mathbf{a}$, Eq.(2-13), and use the substantial derivative and the definition of the mass, $m = \rho \Delta x \Delta y \Delta z$. Then the x-momentum equation becomes {writing Eq.(2-13) as $m\mathbf{a} = \mathbf{F}$, the usual fluid mechanics convention, and considering the x component, $ma_x = F_x$},

$$\rho \Delta x \, \Delta y \, \Delta z \frac{Du}{Dt} = \rho \, \Delta x \, \Delta y \Delta z \, f_x + \frac{\partial}{\partial x}(\tau_{xx}) \Delta x \, \Delta y \Delta z + \frac{\partial}{\partial y}(\tau_{yx}) \Delta y \, \Delta x \Delta z + \frac{\partial}{\partial z}(\tau_{zx}) \Delta y \, \Delta x \, \Delta z . \qquad (2\text{-}27)$$

The $\Delta x \Delta y \Delta z$'s cancel out and can be dropped. The final equations can now be written. Completing the system with the y- and z- equations we obtain,

$$\rho \frac{Du}{Dt} = \rho f_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho \frac{Dv}{Dt} = \rho f_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho \frac{Dw}{Dt} = \rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} . \qquad (2\text{-}28)$$

These are general conservation of momentum relations, valid for anything!

To make Eq. (2-28) specific, we need to relate the stresses to the motion of the fluid. For gases and water, stress is a *linear* function of the rate of strain. Such a fluid is called a Newtonian fluid, *i.e.*:

$$\tau = \mu \frac{\partial u}{\partial y} \qquad (2\text{-}29)$$

where μ is the coefficient of viscosity. In our work we consider μ to be a function of temperature only. Note that in air the viscosity coefficient increases with increasing temperature, and in water the viscosity coefficient decreases with temperature increases.

To complete the specification of the connection between stress and rate of strain, we need to define precisely the relation between the stresses and the motion of the fluid. This can become complicated. In general the fluid description requires two coefficients of viscosity. The coefficient of viscosity arising from the shear stress is well defined. The second coefficient of viscosity is not. This coefficient depends on the normal stress, and is only important in computing the detailed structure of shock waves. Various assumptions relating the coefficients of viscosity are made. The set of assumptions which leads to the equations known as the Navier-Stokes equations are:

- The stress-rate-of-strain relations must be independent of coordinate system.
- When the fluid is at rest and the velocity gradients are zero (the strain rates are zero), the stress reduces to the hydrostatic pressure.
- Stoke's Hypothesis is used to eliminate the issue of mean pressure *vs* thermodynamic pressure (this is the assumption between viscosity coefficients).

Details of the theory associated with these requirements can be found in Schlichting[7] and Grossman.[5] Using the conditions given above leads to the following relations:

$$\tau_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial w}{\partial z} \qquad (2\text{-}30)$$

and

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right). \qquad (2\text{-}31)$$

Combining and neglecting the body force (standard in aerodynamics), we get:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu \nabla \cdot \mathbf{V}\right). \quad (2\text{-}32)$$

These are the classic Navier-Stokes Equations (written in the standard aerodynamics form, which neglects the body force). They are *i*) non-linear {recall that superposition of solutions is not allowed, remember *D/Dt*}, *ii*) highly coupled, and *iii*) long! As written above it's easy to identify ***F*** = *m**a***, written in the fluid mechanics form *m**a*** = ***F***.

When the viscous terms are small, and thus ignored, the flow is termed inviscid.

The resulting equations are known as the *Euler Equations.*

There are also alternate integral formulations of the equations. Consider the momentum flux through an arbitrary control volume in a manner similar to the integral statement of the continuity equation pictured in Fig. 2-4 and given in Eq.(2-7). Here, the momentum change, $\rho\mathbf{V}$, is proportional to the force. The integral statement is:

$$\oint\!\!\oint \rho \mathbf{V}(\mathbf{V}\cdot\hat{n})ds + \frac{\partial}{\partial t}\iiint_v \rho \mathbf{V} dv = \mathbf{F} = \mathbf{F}_{volume} + \mathbf{F}_{surface}. \quad (2\text{-}33)$$

and this statement can also be converted to the differential form using the Gauss Divergence Theorem. Note that we use the derivative notation $\partial / \partial t$ to denote the change in the fixed "porous" control volume that has fluid moving across the boundaries.

The derivation of the Navier-Stokes Equations is for general unsteady fluid motion. Because of limitations in our computational capability (for some time to come), these equations are for laminar flow. When the flow is turbulent, the usual approach is to Reynolds-average the equations, with the result that additional Reynolds stresses appear in the equations. Clearly, the addition of new unknowns requires additional equations. This problem is treated through turbulence modeling and is discussed in Chapter 10, Viscous Effects in Aerodynamics.

2.2.3 The Energy Equation

The equation for the conservation of energy is required to complete the system of equations. This is a statement of the 1st Law of Thermodynamics: *The sum of the work and heat added to a system will equal the increase of energy*. Following the derivation given by White:[8]

$$\underbrace{dE_t}_{\substack{\text{change of total energy}\\ \text{of the system}}} = \underbrace{\delta Q}_{\text{change of heat added}} + \underbrace{\delta W}_{\substack{\text{change of work done}\\ \text{on the system}}}. \tag{2-34}$$

For our fixed control volume coordinate system, the rate of change is:

$$\frac{DE_t}{Dt} = \dot{Q} + \dot{W} \tag{2-35}$$

where:

$$E_t = \rho\left(e + \frac{1}{2}V^2 - \mathbf{g}\cdot\mathbf{r}\right) \tag{2-36}$$

and e is the internal energy per unit mass. The last term is the potential energy, *i.e.* the body force. In aerodynamics this term is neglected. E_t can also be written in terms of specific energy as:

$$E_t = \rho e_{0,} \tag{2-37}$$

where:

$$e_0 = e + \frac{1}{2}V^2 \quad . \tag{2-38}$$

To obtain the energy equation we need to write the RHS of Eq.(2-35) in terms of flow properties. Consider first the heat added to the system.* The heat flow into the control volume is found in the identical manner to the mass flow. Using Fig. 2-8 for reference, obtain the expression for the net heat flow.

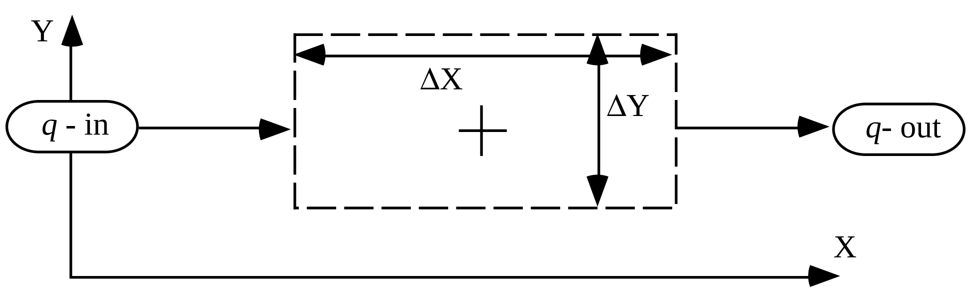

Figure 2-8. *x*-component of heat flux into and out of the control volume.

* Here we neglect heat addition due to radiation. See Grossman[5] for the extension to include this contribution.

The heat fluxes are:

$$q_{x_{in}} = \left(q_x - \frac{\partial q}{\partial x}\frac{\Delta x}{2} \right)\Delta y$$
$$q_{x_{out}} = \left(q_x + \frac{\partial q}{\partial x}\frac{\Delta x}{2} \right)\Delta y \tag{2-39}$$

and the net heat flow into the control volume in the x-direction is $q_{x_{\mathrm{in}}}$ - $q_{x_{\mathrm{out}}}$, or:

$$-\frac{\partial q}{\partial x}\Delta x \Delta y \cdot$$

Similarly, using the same analysis in the *y* and *z* directions we obtain the net heat flux into the control volume (realizing that the $\Delta x \Delta y \Delta z$ terms will cancel):

$$\dot{Q} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) = -\nabla \mathbf{q}. \tag{2-40}$$

Now relate the heat flow to the temperature field. Fourier's Law provides this connection:

$$\mathbf{q} = -k\nabla T \tag{2-41}$$

where *k* is the coefficient of thermal conductivity. Eq.(2-41) is then put into Eq.(2-40) to get the heat conduction in terms of the temperature gradient:

$$\dot{Q} = -\nabla \cdot \mathbf{q} = +\nabla \cdot (k\nabla T). \tag{2-42}$$

Next find the work done on the system. Using the definition of *work* = *force* x *distance*, the *rate* of work is:

$$\dot{W} = force \times velocity. \tag{2-43}$$

Using the control volume again, we find the work, which is equal to the velocity times the stress. The work associated with the *x*-face of the control volume (for two-dimensional flow) is:

$$w_x = u\tau_{xx} + v\tau_{xy}. \tag{2-44}$$

The complete description of the work on the control volume is shown in Figure 2-9.

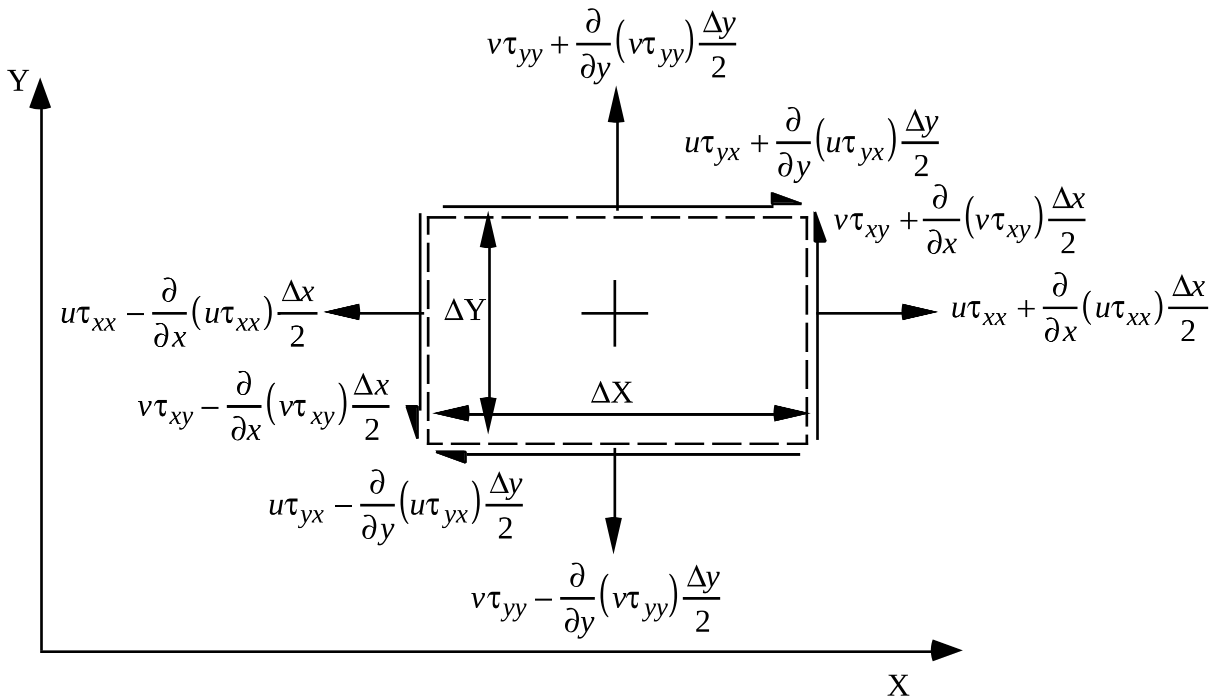

Figure 2-9 Work done on a control volume.

Using the *x*-component of net work as an example again, the work done on the system is $w_{x_{\text{in}}} - w_{x_{\text{out}}}$, or:

$$\left(w_x - \frac{\partial w_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y - \left(w_x + \frac{\partial w_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y = -\frac{\partial w_x}{\partial x}\Delta x \Delta y \,. \tag{2-45}$$

Including the other directions (and dropping the $\Delta x \Delta y \Delta z$ terms, which cancel out)*:

$$\dot{W} = \text{-div}\,\mathbf{w} = \frac{\partial}{\partial x}\left(u\tau_{xx} + v\tau_{xy}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\tau_{yy}\right). \tag{2-46}$$

Substituting Eqs.(2-37) and (2-38) into (2-35) for E_t, Eq.(2-42) for the heat, and Eq.(2-46) for the work, we obtain:

$$\frac{D\rho\left(e + \frac{1}{2}V^2\right)}{Dt} = \nabla \cdot (k\nabla T) - \text{div}\,\mathbf{w}\,\cdot \tag{2-47}$$

* Here we are using White's notation. Realize there is a difference between *W* and **w**.

Many, many equivalent forms of the energy equation are found in the literature. Often the equation is thought of as an equation for the temperature. We now descibe how to obtain one specific form. Substituting in the relations for the τ's in terms of μ and the velocity gradients, Eqs. (2-29) and (2-30), we obtain the following lengthy expression (see Bertin and Smith[2] page 41-45). Making use of the momentum and continuity equations to "simplify" (?), and finally, introducing the definition of enthalpy, $h = e + p/\rho$, we obtain a frequently written form. This is the classical energy equation, which is given as:

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = \underbrace{\nabla \cdot (k \nabla T)}_{\text{heat conduction}} + \underbrace{\Phi}_{\substack{\text{viscous dissipation} \\ \text{(always positive)}}} \tag{2-48}$$

where

$$\Phi = \mu \left\{ 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \right\}. \tag{2-49}$$

The energy equation can be written in numerous forms, and many different but entirely equivalent forms are available. In particular, the energy equation is frequently written in terms of the total enthalpy, H, to good advantage in inviscid and boundary layer flows. A good discussion of the energy equation is also given by White.[8]

There is also an integral form of this equation:

$$\oiint \rho\left(e + V^2/2\right)(\mathbf{V} \cdot \hat{n})\, ds + \frac{\partial}{\partial t} \iiint_v \rho\left(e + V^2/2\right) dv = \dot{Q} + \dot{W}. \tag{2-50}$$

Here again note that we use the derivative notation $\partial / \partial t$ to denote the change in the fixed "porous" control volume that has fluid moving across the boundaries.

2.3 Boundary Conditions

If all flowfields are governed by the same equations, what makes flowfields different? Boundary conditions are the means through which the solution of the governing equations produce differing results for different situations. In computational aerodynamics the specification of boundary conditions constitutes the major part of any effort. Presuming that the flowfield algorithm selected for a particular problem is already developed and tested, the application of the method usually requires the user to specify the boundary conditions.

In general, the aerodynamicist must specify the boundary conditions for a number of different situations. Perhaps the easiest (and most obvious physically) is the condition on the surface. The statement of the boundary conditions is tightly connected to the flowfield model in use. For an inviscid steady flow over a solid surface the statement of the boundary condition is:

$$\mathbf{V}_R \cdot \mathbf{n} = 0 \tag{2-51}$$

which simply says that the difference between the velocity of the component of flow normal to the surface and the surface normal velocity (the relative velocity, $\mathbf{V}_R$) is zero. This simply means that the flow is parallel to the surface, and is known as the non-penetration condition. If **V** is the fluid velocity and $\mathbf{V}_S$ is the surface velocity, then this becomes,

$$(\mathbf{V} - \mathbf{V}_S) \cdot \mathbf{n} = 0 \cdot \tag{2-52}$$

Finally, if the surface is fixed,

$$\mathbf{V} \cdot \mathbf{n} = 0. \tag{2-53}$$

If the flow is viscous the statement becomes even simpler: **V** = 0, the no-slip condition. If the surface is porous, and there is mass flow, the values of the surface velocity must be specified as part of the problem definition. Numerical solutions of the Euler and Navier-Stokes solutions require that other boundary conditions be specified. In particular, conditions on pressure and temperature are required, and will be discussed in later chapters.

As an example, recall that to obtain the unit normal the body is defined (in 2D) in the form $F(x,y) = 0$, the traditional analytic geometry nomenclature. In terms of the usual two-dimensional notation, the body shape is given by $y = f(x)$, which is then written as:

$$F(x,y) = 0 = y - f(x) \tag{2-54}$$

and

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|}. \tag{2-55}$$

Conditions also must be specified away from the body. Commonly this means that at large distances from the body the flowfield must approach the freestream conditions. In numerical computations the question of the farfield boundary condition can become troublesome. How far away is infinity? Exactly how should you specify the farfield boundary condition numerically? How to best handle these issues is the basis for many papers currently appearing in the literature.

Another important use of boundary conditions arises as a means of modeling physics that would be neglected otherwise. When an approximate flowfield model is used, the boundary conditions frequently provide a means of including key elements of the physics in the problem without having to include the physics explicitly. The most famous example of this is the Kutta Condition, wherein the viscous effects at the trailing edge can be accounted for in an inviscid calculation without treating the trailing edge problem explicitly. Karamcheti[1] discuss boundary conditions in more detail.

2.4 Standard Forms and Terminology of Governing Equations

To understand the literature in computational aerodynamics, several other aspects of the terminology must be discussed. This section provides several of these considerations.

2.4.1. Nondimensionalization

The governing equations should be nondimensionalized. Considering fluid mechanics theory, nondimensionalization reveals important similarity parameters. In practice, many different nondimensionalizations are used, and for a particular code, care must be taken to understand exactly what the nondimensionalization is.

Sometimes the dimensional quantities are defined by ( )*'s or $(\tilde{\ })$'s. In other schemes the non- dimensionalized variables are designated by the special symbols. In the example given here, the non-dimensionalized values are denoted by an ( )*. In this system, once the quantities are defined, the *'s are dropped, and the nondimensionalization is understood.

Many different values can be used. We give an example here, and use the the freestream velocity and flow properties, together with the reference length as follows:

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad z^* = \frac{z}{L} \qquad t^* = \frac{tV_\infty}{L}$$

$$u^* = \frac{u}{V_\infty} \qquad v^* = \frac{v}{V_\infty} \qquad w^* = \frac{w}{V_\infty} \qquad p^* = \frac{p}{\rho_\infty V_\infty^2}$$

$$T^* = \frac{T}{T_\infty} \qquad \rho^* = \frac{\rho}{\rho_\infty} \qquad \mu^* = \frac{\mu}{\mu_\infty} \qquad e_o^* = \frac{e_o}{U_\infty^2} \tag{2-56}$$

Each code will have a set of reference nondimensionalizations similar to these. A specific example is given below in Section 2.4.3. Frequently, the speed of sound is used as the reference velocity. Making sure that you understand the nondimensionalization is an important part of applying the codes to aerodynamics problems properly.

2.4.2. Use of divergence form

The classical forms of the governing equations normally given in textbooks usually are not used for computations (as we gave them above). Instead the divergence, or conservation, form* is used. This form is found to be required for reliable numerical calculation. If discontinuities in the flowfield exist, this form must be used to account for discontinuities correctly. It is a way to improve the capability of the differential form of the governing equations. For example, across a shock wave the denity and velocity both jump in value. However, the product of these quantities, the mass flow, is a constant. Thus we can easily see why it is better numerically to work with the product rather than the individual variables. In this section we show how the divergence forms are obtained from the standard classical form. We use the 2D steady *x*-momentum equation as the example:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}. \tag{2-57}$$

This equation is written using the following identities:

$$\frac{\partial \rho u u}{\partial x} = \rho u \frac{\partial u}{\partial x} + u \frac{\partial \rho u}{\partial x} \tag{2-58}$$

or:

$$\rho u \frac{\partial u}{\partial x} = \frac{\partial \left(\rho u^2\right)}{\partial x} - u \frac{\partial \rho u}{\partial x}, \tag{2-59}$$

and similarly with the second term:

$$\frac{\partial \rho v u}{\partial y} = \rho v \frac{\partial u}{\partial y} + u \frac{\partial \rho v}{\partial y} \tag{2-60}$$

or

$$\rho v \frac{\partial u}{\partial y} = \frac{\partial \rho v u}{\partial y} - u \frac{\partial \rho v}{\partial y}. \tag{2-61}$$

* Be careful here, the continuity, momentum and energy equations are all conservation equations. The terminology can be confusing. Conservation *form* refers to the situation where the the variables are inside the derivatives. That's why I prefer the use of divergence form to describe this mathematical arrangement. Conservation form is the more widely used terminalogy. They are both the same.

Substituting (2-59) and (2-61) into (2-57):

$$\frac{\partial \rho u^2}{\partial x} - u\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v u}{\partial y} - u\frac{\partial \rho v}{\partial y} + \frac{\partial p}{\partial x} = 0 \qquad (2\text{-}62)$$

which can be written:

$$\frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho v u}{\partial y} - u\underbrace{\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial x}\right)}_{=0 \text{ from continuity}} + \frac{\partial p}{\partial x} = 0\,. \qquad (2\text{-}63)$$

Finally, the *x*-momentum equation written in divergence form for 2D steady flow is:

$$\frac{\partial\left(\rho u^2 + p\right)}{\partial x} + \frac{\partial\,(\rho v u)}{\partial y} = 0\,. \qquad (2\text{-}64)$$

The equations must be written in divergence form to be valid when shock waves are present.

2.4.3. Standard Form of the Equations

Even after writing the govering equations in divergence form, the equations that you see in the literature won't look like the ones we've been writing down. A standard form is used in the literature for numerical solutions of the Navier-Stokes equations. In this section we provide one representative set. They come from the NASA Langley codes **cfl3d** and **cfl3de**. Professors Walters and Grossman and their students have made contributions to these codes. The Navier-Stokes equations (and the other equations required in the system) are written in vector divergence form as follows:

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial\,(\mathbf{F} - \mathbf{F}_v)}{\partial x} + \frac{\partial\,(\mathbf{G} - \mathbf{G}_v)}{\partial y} + \frac{\partial\,(\mathbf{H} - \mathbf{H}_v)}{\partial z} = 0 \qquad (2\text{-}65)$$

where the conserved variables are:

$$\mathbf{Q} = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_t \end{Bmatrix} = \begin{Bmatrix} \text{density} \\ \text{x - momentum} \\ \text{y - momentum} \\ \text{z - momentum} \\ \text{total energy per unit volume} \end{Bmatrix}. \qquad (2\text{-}66)$$

The flux vectors in the *x*-direction are:

Inviscid terms Viscous terms

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E_t + p)u \end{bmatrix} \qquad \mathbf{F}_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - \dot{q}_x \end{bmatrix}. \tag{2-67}$$

Similar expressions can be written down for the *y*- and *z*-direction fluxes, with the *y*-direction given as:

Inviscid terms Viscous terms

$$\mathbf{G} = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vw \\ (E_t + p)v \end{bmatrix} \qquad \mathbf{G}_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - \dot{q}_y \end{bmatrix}, \tag{2-68}$$

and in the *z*-direction:

Inviscid terms Viscous terms

$$\mathbf{H} = \begin{bmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ (E_t + p)w \end{bmatrix} \qquad \mathbf{H}_v = \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - \dot{q}_z \end{bmatrix}. \tag{2-69}$$

The equation of state (perfect gas) is written in this formulation as:

$$p = (\gamma - 1)\left[E_t - \rho\left(u^2 + v^2 + w^2\right)/2\right]. \tag{2-70}$$

To complete the flow equations, we need to define the nondimensionalization, and the shear stress and heat transfer nomenclature.

Shear stress and heat transfer terms are written in indicial (or index*) notation as:

$$\tau_{x_i x_j} = \frac{M_\infty}{\mathrm{Re}_L}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda \frac{\partial u_k}{\partial x_k}\delta_{ij}\right] \tag{2-71}$$

* Index notation is a shorthand notation. x_i denotes *x,y,z* for i = 1,2,3.

and:

$$\dot{q}_{x_i} = -\left[\frac{M_\infty \mu}{\mathrm{Re}_L \Pr(\gamma - 1)}\right]\frac{\partial\left(a^2\right)}{\partial x_i} = -\left[\frac{M_\infty \mu}{\mathrm{Re}_L \Pr(\gamma - 1)}\right]\frac{\partial T}{\partial x_i}. \tag{2-72}$$

The molecular viscosity is found using Sutherland’s Law:

$$\mu = \tilde{\mu} / \tilde{\mu}_\infty = \left(\frac{\tilde{T}}{\tilde{T}_\infty}\right)^{3/2}\left[\frac{\tilde{T}_\infty + \tilde{c}}{\tilde{T} + \tilde{c}}\right] = (T)^{3/2}\left[\frac{\left(1 + \tilde{c} / \tilde{T}_\infty\right)}{\left(T + \tilde{c} / \tilde{T}_\infty\right)}\right] \tag{2-73}$$

where Sutherland’s constant is $\dot{c} = 198.6°\mathrm{R} = 110.4°\mathrm{K}$. The tilde, ($\tilde{\ }$), superscript denotes a dimensional quantity and the subscript infinity denotes evaluation at freestream conditions. The other quantities are defined as: Reynolds number, $\mathrm{Re}_L = \tilde{\rho}_\infty \tilde{q}_\infty \tilde{L} / \tilde{\mu}_\infty$, Mach number, $M_\infty = \tilde{q}_\infty / \tilde{a}_\infty$, and Prandtl number, $\Pr = \tilde{\mu}\,\tilde{c}_p / \tilde{k}$. Stoke’s hypothesis for bulk viscosity is used, meaning $\lambda + 2\mu / 3 = 0$, and the freestream velocity magnitude is, $\tilde{q}_\infty = \left[\tilde{u}_\infty^2 + \tilde{v}_\infty^2 + \tilde{w}_\infty^2\right]^{1/2}$.

The velocity components are given by:

$$\begin{aligned} u &= \tilde{u} / \tilde{a}_\infty & u_\infty &= M_\infty \cos\alpha \cos\beta \\ v &= \tilde{v} / \tilde{a}_\infty & v_\infty &= -M_\infty \sin\beta \\ w &= \tilde{w} / \tilde{a}_\infty & w_\infty &= M_\infty \sin\alpha \cos\beta \end{aligned} \tag{2-74}$$

and the thermodynamic variables are given by:

$$\begin{aligned} \rho &= \tilde{\rho} / \tilde{\rho}_\infty, & \rho_\infty &= 1 \\ p &= \tilde{p} / \tilde{\rho}\tilde{a}_\infty^2, & p_\infty &= 1/\gamma \\ T &= \tilde{T} / \tilde{T}_\infty = \gamma p / \rho = a^2 & T_\infty &= 1 \end{aligned} \tag{2-75}$$

and,

$$E_t = \tilde{E}_t / \tilde{\rho}_\infty \tilde{a}_\infty^2 \qquad E_{t_\infty} = 1/\left[\gamma(\gamma - 1)\right] + M_\infty^2 / 2. \tag{2-76}$$

This completes the nomenclature for one typical example of the application of the Navier-Stokes equations in an actual current computer code. Note that these equations are for a Cartesian coordinate system. We will discuss the necessary extension to general coordinate systems in the Chapter 9, Geometry and Grids: Major Considerations Using Computational Aerodynamics.

2.5 The Gas Dynamics Equation and the Full Potential Equation

For inviscid flow (and even some viscous flow problems) it is useful to combine the equations in a special form known as the gas dynamics equation. In particular, this equation is used to obtain the complete or "full" nonlinear potential flow equation. Many valuable results can be obtained in computational aerodynamics (CA) using the potential flow approximation. When compressibility effects are important, a special form of the governing equation can be obtained. This equation is based on the so-called gas dynamics equation, which we derive here. The gas dynamics equation is valid for any flow assumed to be inviscid. The starting point for the derivation is the Euler equations, the continuity equation and the equation of state.

2.5.1 The Gas Dynamics Equation

We demonstrate the derivation using two-dimensional steady flow. (This is not required. Furthermore, the notation x_i , which is known as index notation, denotes x,y,z for i = 1,2,3). To start, we make use of a thermodynamic definition to rewrite the pressure term in the momentum equation.

$$\frac{\partial p}{\partial x_i} = \left.\frac{\partial p}{\partial \rho}\right)_s \frac{\partial \rho}{\partial x_i} \tag{2-77}$$

and recall the definition of the speed of sound:

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right)_s \tag{2-78}$$

allowing $\partial p/\partial x_i$ to be written as:

$$\frac{\partial p}{\partial x_i} = a^2 \frac{\partial \rho}{\partial x_i}. \tag{2-79}$$

We next write u times the x and v times the y momentum equations:

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} &= -\frac{u}{\rho}\frac{\partial p}{\partial x} = -u\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \\ vu\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} &= -\frac{v}{\rho}\frac{\partial p}{\partial y} = -v\frac{a^2}{\rho}\frac{\partial \rho}{\partial y} \end{aligned} \tag{2-80}$$

and use the continuity equation by expanding it from

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{2-81}$$

to

$$u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial v}{\partial y} = 0 \tag{2-82}$$

or

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = -\rho\frac{\partial u}{\partial x} - \rho\frac{\partial v}{\partial y}. \tag{2-83}$$

Now add the modified *x*- and *y*- momentum equations given above:

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} + vu\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} &= -u\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} - v\frac{a^2}{\rho}\frac{\partial \rho}{\partial y} \\ &= -\frac{a^2}{\rho}\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right). \end{aligned} \tag{2-84}$$

Substitute into this equation the rewritten continuity equation from above:

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} + vu\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} &= -\frac{a^2}{\rho}\left(-\rho\frac{\partial u}{\partial x} - \rho\frac{\partial v}{\partial y}\right) \\ &= a^2\frac{\partial u}{\partial x} + a^2\frac{\partial v}{\partial y}. \end{aligned} \tag{2-85}$$

Finally, collecting terms we obtain in two dimensions:

$$(u^2 - a^2)\frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (v^2 - a^2)\frac{\partial v}{\partial y} = 0 \tag{2-86}$$

or in three dimensions:

$$(u^2 - a^2)\frac{\partial u}{\partial x} + (w^2 - a^2)\frac{\partial w}{\partial z} + (v^2 - a^2)\frac{\partial v}{\partial y} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + vw\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + wu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \tag{2-87}$$

This equation is known as the gas dynamics equation.

2.5.2 Derivation of the Classical Gas Dynamics-Related Energy Equation

The special form of the energy equation that is used to close the system is given by (in 2D):

$$a^2 = a_0^2 - \left(\frac{\gamma - 1}{2}\right)\left(u^2 + v^2\right) \tag{2-88}$$

and we need to show exactly how this relation is obtained. Start with the form of the energy equation for inviscid, adiabatic flow:

$$\frac{DH}{Dt} = 0 \tag{2-89}$$

which yields H = constant, where H is (in two dimensions) the total enthalpy, defined by:

$$H = h + \frac{1}{2}\left(u^2 + v^2\right). \tag{2-90}$$

Thus we have a purely algebraic statement of the energy equation instead of a partial differential equation. This is an important reduction in complexity.

For a thermally and calorically perfect gas, $h = c_p T$, and c_p = constant. Substituting for the enthalpy, we get

$$c_p T_0 = c_p T + \frac{1}{2}\left(u^2 + v^2\right). \tag{2-91}$$

Recalling that $a^2 = \gamma RT$ and $R = c_p - c_v$, with $\gamma = c_p/c_v$, we write

$$a^2 = \frac{c_p}{c_v}\left(c_p - c_v\right)T = \left(\frac{c_p - c_v}{c_v}\right)c_p T \tag{2-92}$$

or:

$$c_p T = \left(\frac{c_v}{c_p - c_v}\right)a^2 = \left(\frac{1}{\gamma - 1}\right)a^2 \tag{2-93}$$

and substitute into the total energy equation (H = constant), Eqn. (2-91),

$$\frac{a_0^2}{\gamma - 1} = \frac{a^2}{\gamma - 1} + \frac{1}{2}\left(u^2 + v^2\right) \tag{2-94}$$

or:

$$a_0^2 = a^2 + \left(\frac{\gamma - 1}{2}\right)\left(u^2 + v^2\right) \tag{2-95}$$

and finally, solving for a (and including the third dimension):

$$a^2 = a_0^2 - \left(\frac{\gamma - 1}{2}\right)(u^2 + v^2 + w^2) \tag{2-96}$$

which is the equation we have been working to find.

2.5.3 Full Potential Equation

The gas dynamics equation is converted to the classical nonlinear potential equation when we make the irrotational flow assumption. The potential flow assumption requires that the flow be irrotational. This is valid for inviscid flow when the onset flow is uniform and there are no shock waves. However, we often continue to assume the flow can be represented approximately by a potential when the Mach number normal to any shock wave is close to one ($M_n < 1.25$, say). Recall that the irrotational flow assumption is stated mathematically as curl **V**=0. When this is true, **V** can be defined as the gradient of a scalar quantity, $\mathbf{V} = \nabla\Phi$. Using the common subscript notation to represent partial derivatives, the velocity components are $u = \Phi_x$, $v = \Phi_y$ and $w = \Phi_z$. Using the gas dynamics equation, the non-linear or "full" potential equation is then:

$$(\Phi_x^2 - a^2)\Phi_{xx} + (\Phi_y^2 - a^2)\Phi_{yy} + (\Phi_z^2 - a^2)\Phi_{zz} + 2\Phi_x\Phi_y\Phi_{xy} + 2\Phi_y\Phi_z\Phi_{yz} + 2\Phi_z\Phi_x\Phi_{zx} = 0 \,. \tag{2-97}$$

This is the classic form of the equation. It has been used for many years to obtain physical insight into a wide variety of flows. This is a single partial differential equation. However, it is a nonlinear equation, and as written above, it is not in divergence (or conservation) form.

2.5.4 Equivalent Divergence Form and Energy Equation

The equivalent equation written in conservation form makes use of the continuity equation. This is the form that is used in most computational fluid dynamics codes. Written here in two dimensions it is:

$$\frac{\partial}{\partial x}(\rho\Phi_x) + \frac{\partial}{\partial y}(\rho\Phi_y) = 0. \tag{2-98}$$

The relation between ρ and the potential is given by:

$$\rho = \left[1 - \left(\tfrac{\gamma-1}{\gamma+1}\right)(\Phi_x^2 + \Phi_y^2)\right]^{\frac{1}{\gamma-1}} \tag{2-99}$$

which is a statement of the energy equation. Note that the full potential equation is still nonlinear when the density varies and ρ must be considered a dependent variable.

2.5.5 Derivation of another form of the Related Energy Equation

It is informative to demonstrate the derivation of the energy equation given above. To get this standard form, understand the specific non-dimensionalization employed with this form:

$$\rho = \frac{\tilde{\rho}}{\rho_0}, \qquad \Phi_x = \frac{\tilde{u}}{a^*}, \qquad \Phi_y = \frac{\tilde{v}}{a^*}, \tag{2-100}$$

where a^* denotes the sonic value. Start with the previous energy equation and work with dimensional variables for the moment:

$$a^2 = a_0^2 - \left(\frac{\gamma - 1}{2}\right)(u^2 + v^2) \tag{2-101}$$

or

$$\frac{a^2}{a_0^2} = 1 - \left(\frac{\gamma - 1}{2}\right)\left(\frac{u^2 + v^2}{a_0^2}\right). \tag{2-102}$$

Now, get a relation for a_o in terms of the eventual nondimensionalizing velocity a^*:

$$a_0^2 = a^2 + \left(\frac{\gamma - 1}{2}\right)\underbrace{(u^2 + v^2)}_{= a^2} \tag{2-103}$$

when the velocity is equal to the speed of sound $a = a^*$. Combining terms:

$$a_0^2 = a^{*2} + \left(\frac{\gamma - 1}{2}\right)a^{*2} = \left(1 + \frac{\gamma - 1}{2}\right)a^{*2} \tag{2-104}$$

or:

$$a_0^2 = \left(\frac{\gamma + 1}{2}\right)a^{*2}. \tag{2-105}$$

Replace a_o^2 in the energy relation with a^{*2} in the velocity term (denominator of Eq. 2-102). And in the first term use:

$$\left(\frac{a}{a_0}\right)^2 = \frac{T}{T_0}, \tag{2-106}$$

recalling $a^2 = \gamma RT$, to get:

$$\frac{T}{T_0} = 1 - \frac{\gamma - 1}{2}\frac{u^2 + v^2}{\frac{\gamma + 1}{2}a^{*2}} \tag{2-107}$$

or

$$\frac{T}{T_0} = 1 - \left(\frac{\gamma - 1}{\gamma + 1}\right)\left[\left(\frac{u}{a^*}\right)^2 + \left(\frac{v}{a^*}\right)^2\right]. \tag{2-108}$$

Recall for isentropic flow (a consistent assumption if the use of Φ is valid):

$$s = const \tag{2-109}$$

and

$$\frac{p}{\rho^\gamma} = const = \frac{p_0}{\rho_0^\gamma} \tag{2-110}$$

Now, we introduce ($\tilde{}$) to denote dimensional quantities and convert to the desired nondimensional form:

$$\left(\frac{\tilde{p}}{p_0}\right) = \left(\frac{\tilde{\rho}}{\rho_0}\right)^\gamma = \left(\frac{\tilde{T}}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \tag{2-111}$$

or

$$\frac{\tilde{T}}{T_0} = \left(\frac{\tilde{\rho}}{\rho_0}\right)^{\gamma - 1} \tag{2-112}$$

Using Eq. (2-112) we write the energy equation, Eq. (2-108), as:

$$\left(\frac{\tilde{\rho}}{\rho_0}\right)^{\gamma-1} = 1 - \left(\frac{\gamma - 1}{\gamma + 1}\right)\left[\left(\frac{\tilde{u}}{a^*}\right)^2 + \left(\frac{\tilde{v}}{a^*}\right)^2\right] \tag{2-113}$$

Using the nondimensionalizing definition given above, we finally obtain:

$$\rho = \left[1 - \left(\tfrac{\gamma-1}{\gamma+1}\right)\left(\Phi_x^2 + \Phi_y^2\right)\right]^{\frac{1}{\gamma-1}} \tag{2-114}$$

This is an energy equation in ρ to use with the divergence form of the full potential equation. It is also an example of how to get an energy equation in a typical nondimensional form used in the literature.

2.6 Special Cases

In this section we present a number of special, simplified forms of the equations described above. These simplified equations are entirely adequate for many of the problems of computational aerodynamics, and until recently were used nearly exclusively. The ability to obtain simpler relations, which provide explicit physical insight into the flowfield process, has played an important role in the development of aerodynamic concepts. One key idea is the notion of small disturbance equations. The assumption is that the flowfield is only slightly disturbed by the body. We expect this assumption to be valid for inviscid flows over streamlined shapes. These ideas are expressed mathematically by small perturbation or asymptotic expansion methods, and are elegantly described in the book by Van Dyke.[9] The figure at the end of this section summarizes the theoretical path required to obtain these equations.

2.6.1 Small Disturbance Form of the Energy Equation

The expansion of the simple algebraic statement of the energy equation provides an example of a small disturbance analysis. In this case the square of the speed of sound (or equivalently the temperature) is linearly related to the velocity field. Start with the energy equation:

$$a^2 = a_0^2 - \left(\frac{\gamma - 1}{2}\right)\left(u^2 + v^2\right) \tag{2-115}$$

and

$$a_0^2 = const = a^2 + \left(\frac{\gamma - 1}{2}\right)\left(u^2 + v^2\right) = a_\infty^2 + \frac{\gamma - 1}{2} U_\infty^2 . \tag{2-116}$$

Letting $u = U_\infty + u'$, $v = v'$:

$$a^2 = a_\infty^2 + \frac{\gamma - 1}{2} U_\infty^2 - \left(\frac{\gamma - 1}{2}\right)\left[U_\infty^2 + 2U_\infty u' + u'^2 + v'^2\right] \tag{2-117}$$

and combining terms:

$$a^2 = a_\infty^2 - \left(\frac{\gamma - 1}{2}\right)\left[2U_\infty u' + u'^2 + v'^2\right]. \tag{2-118}$$

At this point the relation is still exact, but now it is written so that it can easily be simplified. The basic idea will be to take advantage of the assumption:

$$u' < U_\infty, \qquad v' < U_\infty \tag{2-119}$$

and thus,

$$\frac{u'}{U_\infty} < 1 \quad \Rightarrow \left(\frac{u'}{U_\infty}\right)^2 \approx 0 \tag{2-120}$$

where the above equation becomes:

$$a^2 = a_\infty^2 - \left(\frac{\gamma - 1}{2}\right)[2U_\infty u' + \underbrace{u'^2 + v'^2}_{\text{neglect as small henceforth}}]. \tag{2-121}$$

This is a linear relation between the disturbance velocity and the speed of sound. It is a heuristic example of the procedures used in a more formal approach known as perturbation theory.

2.6.2 Small Disturbance Expansion of the Full Potential Equation

We now use a similar approach to show how to obtain a small disturbance version of the full potential equation. Again consider the situation where we assume that the disturbance to the freestream is small. Now we examine the full potential equation. First, we rewrite the full potential equation given above (in 2D for simplicity):

$$(\Phi_x^2 - a^2)\Phi_{xx} + 2\Phi_x\Phi_y\Phi_{xy} + (\Phi_y^2 - a^2)\Phi_{yy} = 0 \cdot \tag{2-122}$$

Now write the velocity as a difference from the freestream velocity. Introduce a disturbance potential ϕ, defined by:

$$\begin{aligned} \Phi &= U_\infty x + \phi(x, y) \\ \Phi_x &= u = U_\infty + \phi_x \\ \Phi_y &= v = \phi_y \end{aligned} \tag{2-123}$$

where we have introduced a *directional bias*. The x- direction is the direction of the freestream velocity. We will assume that ϕ_x and ϕ_y are small compared to U_∞. Using the idea of a small disturbance to the freestream, simplified (and even linear) forms of a small disturbance potential equation and an energy equation can be derived.

As an example of the expansion process, consider the first term. Use the definition of the disturbance potential and the simplified energy equation as:

$$
\begin{aligned}
(\Phi_x^2 - a^2) &\cong (U_\infty + \phi_x)^2 - \left\{ a_\infty^2 - \left(\frac{\gamma - 1}{2}\right)[2U_\infty u'] \right\} \\
&\cong U_\infty^2 + 2U_\infty \phi_x + \phi_x^2 - a_\infty^2 + \frac{\gamma - 1}{2} 2U_\infty \underbrace{u'}_{=\phi_x} .
\end{aligned}
\tag{2-124}
$$

Regroup and drop the square of the disturbance velocity as small:

$$
\begin{aligned}
(\Phi_x^2 - a^2) &\cong U_\infty^2 - a_\infty^2 + 2U_\infty \phi_x + (\gamma - 1)U_\infty \phi_x \\
&\cong U_\infty^2 - a_\infty^2 + \underbrace{[2 + (\gamma - 1)]}_{\gamma + 1} U_\infty \phi_x \\
&\cong U_\infty^2 - a_\infty^2 + (\gamma + 1)U_\infty \phi_x .
\end{aligned}
\tag{2-125}
$$

Dividing by a_∞^2

$$
\begin{aligned}
\left(\frac{\Phi_x^2}{a_\infty^2} - \frac{a^2}{a_\infty^2}\right) &\cong \frac{U_\infty^2}{a_\infty^2} - 1 + (\gamma + 1)\frac{U_\infty}{a_\infty}\frac{\phi_x}{a_\infty} \\
&\cong M_\infty^2 - 1 + (\gamma + 1)M_\infty \underbrace{\frac{U_\infty}{U_\infty}\frac{\phi_x}{a_\infty}}_{\frac{U_\infty}{a_\infty}\frac{\phi_x}{U_\infty}} \\
&\cong \left(M_\infty^2 - 1\right) + (\gamma + 1)M_\infty^2 \left(\frac{\phi_x}{U_\infty}\right).
\end{aligned}
\tag{2-126}
$$

Rewrite the potential equation, Eq. (2-122) dividing by a_∞^2. Then replace the coefficient of the first term using Eq. (2-126):

$$
\underbrace{\left(\frac{\Phi_x^2}{a_\infty^2} - \frac{a^2}{a_\infty^2}\right)}_{\left[\left(M_\infty^2 - 1\right) + (\gamma + 1)M_\infty^2\left(\frac{\phi_x}{U_\infty}\right)\right]} \Phi_{xx} + 2\frac{\Phi_x}{a_\infty}\frac{\Phi_y}{a_\infty}\Phi_{xy} + \left(\frac{\Phi_y^2}{a_\infty^2} - \frac{a^2}{a_\infty^2}\right)\Phi_{yy} = 0 .
\tag{2-127}
$$

Now, by definition

$$\Phi_{xx} = \phi_{xx}, \quad \Phi_{yy} = \phi_{yy}, \quad \Phi_{xy} = \phi_{xy} \tag{2-128}$$

while:

$$\frac{\Phi_x}{a_\infty} = M_\infty\left(1 + \frac{\phi_x}{U_\infty}\right) \qquad \frac{\Phi_y}{a_\infty} = M_\infty \frac{\phi_y}{U_\infty} \tag{2-129}$$

and using the same approach demonstrated above we can write:

$$\left(\frac{\Phi_y^2}{a_\infty^2} - \frac{a^2}{a_\infty^2}\right) \cong -1 + (\gamma - 1) M_\infty^2 \left(\frac{\phi_y}{U_\infty}\right). \tag{2-130}$$

Putting these relations all into the potential equation we obtain:

$$\left[M_\infty^2 - 1 + (\gamma + 1) M_\infty^2 \frac{\phi_x}{U_\infty}\right]\phi_{xx} + 2M_\infty^2\left(1 + \frac{\phi_x}{U_\infty}\right)\frac{\phi_y}{U_\infty}\phi_{xy} + \left[-1 + (\gamma - 1) M_\infty^2 \frac{\phi_y}{U_\infty}\right]\phi_{yy} = 0 \tag{2-131}$$

where the ϕ_x^2, ϕ_y^2 terms are neglected in the coefficients. This equation is still nonlinear, but is in a form ready for the further simplifications described below.

2.6.3 Transonic Small Disturbance Equation

Transonic flows contain regions with both subsonic and supersonic velocities. Any equation describing this flow must simulate the correct physics in the two different flow regimes. As we will show below, this makes the problem difficult to solve numerically. Indeed, the numerical solution of transonic flows was one of the primary thrusts of research in CFD over the decades of the ’70s and ’80s. A small disturbance equation can be derived that captures the essential nonlinearity of transonic flow, which is the rapid streamwise variation of flow disturbances in the *x*-direction, including normal shock waves. Therefore, in transonic flows:

$$\frac{\partial}{\partial x} > \frac{\partial}{\partial y}. \tag{2-132}$$

The transonic small disturbance equation retains the key term in the convective derivative, $u(\partial u/\partial x)$, which allows the shock to occur in the solution. Retaining this key nonlinear term the small disturbance equation given above becomes:

$$\left[\left(1-M_\infty^2\right)-(\gamma+1)M_\infty^2\frac{\phi_x}{U_\infty}\right]\phi_{xx}+\phi_{yy}=0\,. \tag{2-133}$$

Note that using the definition of the potential from Eq.(2-123) we can identify the nonlinear term, $u(\partial u/\partial x)$, which appears as the product of the second term in the bracket, $u=\phi_x$, and the ϕ_{xx} term, which is $\partial u/\partial x$.

This is one version of the *transonic small disturbance equation*. It is still nonlinear, and can change mathematical *type* (to be discussed in section 2.8). This means that the sign of the coefficient of ϕ_{xx} can change in the flowfield, depending on the value of the nonlinear term. It is valid for transonic flow, and, as written, it is not in a divergence form. Transonic flows occur for Mach numbers from .6 to 1.2, depending on the degree of flow disturbance. They also occur under other circumstances. At high-lift conditions, the flow around the leading edge may become locally supersonic at freestream Mach numbers as low as .20 or .25. Transonic flow occurs on rotor blades and propellers. At hypersonic speeds the flow between the bow shock and the body will frequently be locally subsonic. These are also transonic flows. The transonic small disturbance equation can be solved on your personal computer.

2.6.4 Prandtl-Glauert Equation

When the flowfield is entirely subsonic or supersonic, all terms involving products of small quantities can be neglected in the small disturbance equation. When this is done we obtain the Prandtl-Glauert Equation:

$$\left(1-M_\infty^2\right)\phi_{xx}+\phi_{yy}=0\ . \tag{2-134}$$

This is a linear equation valid for small disturbance flows that are either entirely supersonic or subsonic. For subsonic flows this equation can be transformed to Laplace's Equation, while at supersonic speeds this equation takes the form of a wave equation. The difference is important, as described below in the section on the mathematical type of partial differential equations (PDEs). This equation requires that the onset flow be in the *x*-direction, an example of the importance that coordinate systems assume when simplifying assumptions are made. Thus, use of simplifying assumptions introduced a directional bias into the resulting approximate equation.

The extension to three dimensions is:

$$\left(1-M_\infty^2\right)\phi_{xx}+\phi_{yy}+\phi_{zz}=0\,. \tag{2-135}$$

2.6.5. Incompressible irrotational flow: Laplace's Equation

Assuming that the flow is incompressible, ρ is a constant and can be removed from the modified continuity equation, Eq.(2-97), given above. Alternately, divide the full potential equation by the speed of sound, *a*, squared, and take the limit as *a* goes to infinity. Either way, the following equation is obtained:

$$\phi_{xx}+\phi_{yy}=0\,. \tag{2-136}$$

This is Laplace's Equation. Frequently people call this equation the potential equation. For that reason the complete potential equation given above is known as the *full potential equation*. Do not confuse the true potential flow equation with Laplace's equation, which requires the assumption of incompressible flow. When the flow is incompressible, this equation is exact when using the inviscid irrotational flow model, and does not require the assumption of small disturbances.

2.6.6 The Boundary Layer Equations

The last special case retains a viscous term, while assuming that the pressure is a known function and independent of the *y*-coordinate value. These are the Prandtl boundary layer equations that describe the flow immediately adjacent to the body surface. For 2D, steady flow they are:

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0 \tag{2-137}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2-138}$$

$$0=-\frac{\partial p}{\partial y}\,. \tag{2-139}$$

The related energy equation must also be included if compressibility effects are important.

All the equations presented in this section provide physical models of classes of flows that, under the right circumstances, are completely adequate to obtain an accurate representation of the flow. Many, many other approximate flow models have been proposed. Those presented in this section represent by far the majority of methods currently used. In recent times, numerous versions of the Navier-Stokes equations (taken here to include the time-averaged Reynolds equations to be discussed in Chap. 10) have also been used. These equations will be discussed as appropriate in subsequent chapters. Figure 2-10 given below summarizes the connection between the various flowfield models.

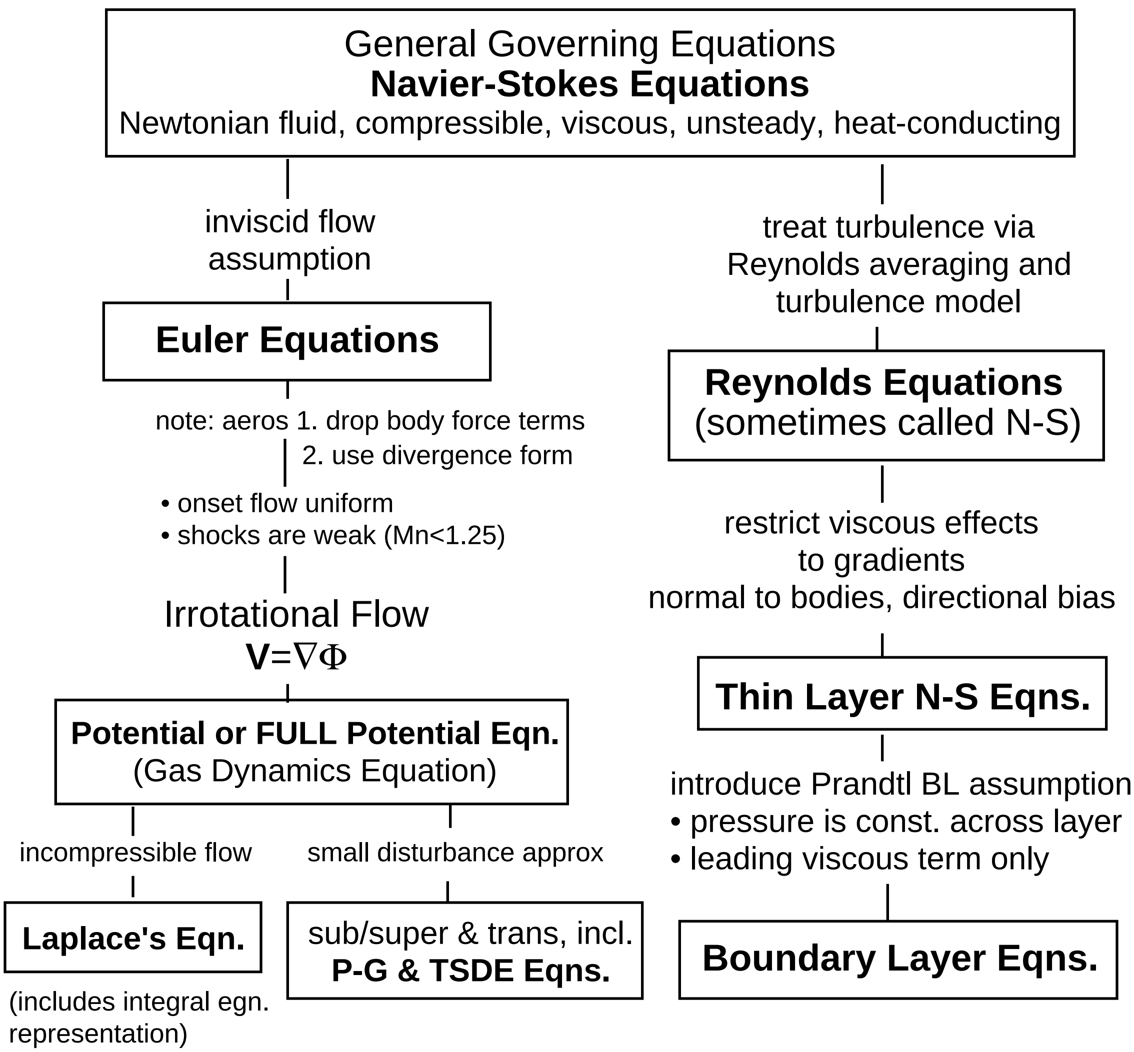

Figure 2-10. Connection between various approximations to the governing equations.

2.7 Examples of Zones of Application

The appropriate version of the governing equation depends on the type of flowfield being investigated. For high Reynolds number attached flow, the pressure can be obtained very accurately without considering viscosity. Recall that the use of a Kutta condition provides a simple way of enforcing key physics associated with viscosity by specifying this feature as a boundary condition on an otherwise inviscid solution. If the onset flow is uniform, and any shocks are weak, $M_n < 1.25$ or 1.3, then the potential flow approximation is valid. If a slight flow separation exists, a special approach using the boundary layer equations can be used interactively with the inviscid solution to obtain a solution. As speed increases, shocks begin to get strong and are curved. Under these circumstances the solution of the complete Euler equations is required.

When significant separation occurs, or you cannot figure out the preferred direction to apply a boundary layer approach, the Navier-Stokes equations are used. Note that many different “levels” of the N-S Equations are in use.

To avoid having many different codes, some people would like to have just one code that does everything. While this is a goal, most applications are better treated using a variety of methods. A step in the right direction is the use of a system that employs a common geometry and grid processing system, and a common output/graphics systems.

2.8 Mathematical Classification or the "Type" of Partial Differential Equations (PDEs)

A key property of any system of PDEs is the “type” of the equations. In mathematics, an equation “type” has a very precise meaning. Essentially, the *type* of the equation determines the domain on which boundary or initial conditions must be specified. The mathematical theory has been developed over a number of years for PDEs, and is given in books on PDEs. Two examples include Sneddon[10] (pages 105-109), and Chester[11] (chapter 6). Discussions from the computational fluid dynamics viewpoint are available in Anderson, Tannehill, and Pletcher[12] (chapter 2), Fletcher[13] (chapter 2), and Hoffman[14] (chapter 1).

To successfully obtain the numerical solution of a PDE you must satisfy the “spirit” of the theory for the type of a PDE. Usually the theory has been developed for model problems, frequently linear. For PDEs describing physical systems, the type will be related to the following categorization:

1. *Equilibrium problems.* Examples include steady state temperature distributions and steady incompressible flow. These are similar to boundary value problems for ordinary differential equations.

2. *Marching or Propagation Problems*. These are transient or transient-like problems. Examples include transient heat conduction and steady supersonic flow. These are similar to initial value problems for ODEs.

The *types* are elliptic, parabolic, and hyperbolic. A linear equation will have a constant type. The nonlinear equations of fluid flow can change type locally depending on the local values of the equation. This “mixed-type” feature had a profound influence on the development of methods for computational aerodynamics. A mismatch between the *type* of the PDE and the prescribed boundary conditions dooms any attempt at numerical solution to failure.

The standard mathematical illustration of *type* uses a second order PDE:

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0 . \qquad (2\text{-}140)$$

where A, B, C, D, E, F, and G can be constants or functions of x, y, and ϕ. Depending on the values of A, B, and C, the PDE will be of different type. The specific type of the PDE depends on the characteristics of the PDE. One of the important properties of characteristics is that the second derivative of the dependent variables are allowed, although there can be no discontinuity of the first derivative. The slopes of the characteristics can be found from A, B, and C. From mathematical theory the characteristics are found depending on the sign of determinant:

		Characteristics	Type	
$(B^2 - 4AC)$	> 0	real	hyperbolic	
	$= 0$	real, equal	parabolic	(2-141)
	< 0	imaginary	elliptic	

Hyperbolic: The basic property is a limited domain of dependence. Initial data are required on a curve C, which does not coincide with a characteristic curve. Figure 2-11 illustrates this requirement.

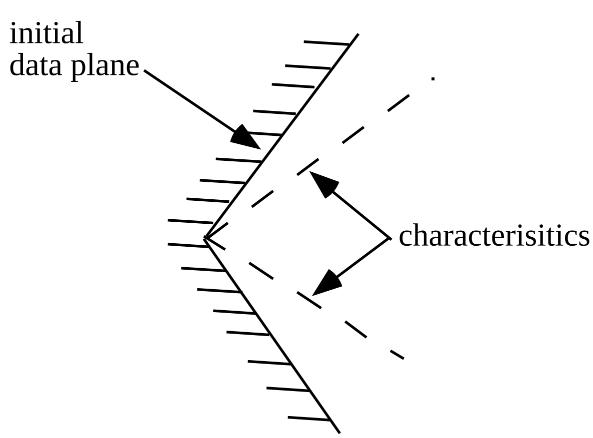

Figure 2-11. Connection between characteristics and initial condition data planes.

Classical linearized supersonic aerodynamic theory is an example of a hyperbolic system.

Parabolic: This is associated with a diffusion process. Data must be specified at an initial plane, and march forward in a time or time-like direction. There is no limited zone of influence equivalent to the hyperbolic case. Data are required on the entire time-like surface. Figure 2-12 illustrates the requirement.

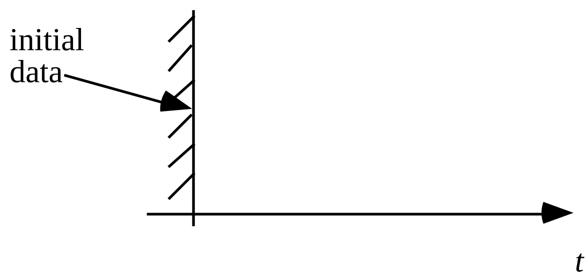

Figure 2-12. Initial data plane for parabolic equation.

In aerodynamics, boundary layers have a parabolic type.

Elliptic: These are equilibrium problems. They require boundary conditions everywhere, as shown in Figure 2-13. Incompressible potential flow is an example of a governing equation of elliptic type.

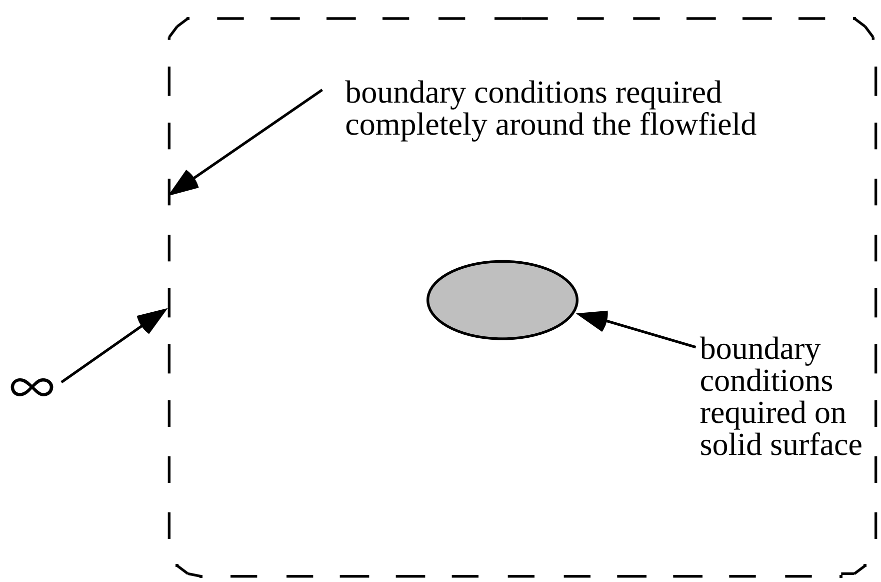

Figure 2-13. Boundary conditions required for elliptic PDEs.

Consider the following examples. For the Prandtl-Glauert equation:

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} = 0 \tag{2-142}$$

and:

$$\begin{aligned} M_\infty &< 1 \quad \text{elliptic} \\ &> 1 \quad \text{hyperbolic}. \end{aligned} \tag{2-143}$$

For the transonic small disturbance equation:

$$\underbrace{\left[\left(1 - M_\infty^2\right) - (\gamma + 1) M_\infty^2 \frac{\phi_x}{U_\infty}\right]}_{\substack{\text{sign depends on the solution} \\ \text{- locally subsonic: elliptic} \\ \text{- locally supersonic: hyperbolic}}} \phi_{xx} + \phi_{yy} = 0 \quad . \tag{2-144}$$

This is an equation of mixed type. It is required to treat the physics of transonic flows.

Type plays a key role in computational approaches. The type can be used to advantage. In the case of the Euler equations, the steady state Euler equations are hard to solve. It is standard procedure to consider the unsteady case, which is hyperbolic, and obtain the steady state solution by marching in time until the solution is constant in time.

Alternate approaches are available for systems of first order PDEs. Classification is sometimes difficult to determine. The *type* of an equation is determined with respect to a particular variable. The *type* of equations with respect to time may be completely different than their type with respect to space. The *type* of the equation often helps to define the appropriate solution coordinate system. The different types of the equations given above are responsible for the distinct numerical approaches that are adopted to solve different problems.

2.8.1 Elaboration on Characteristics

This section provides additional details that provide some insight into the reason that the determinant of the coefficients of the second derivative terms define the type of the equation.

Considering:

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0 \tag{2-145}$$

- Assume ϕ is a solution describing a curve in space
- These curves “patch” various solutions, known as characteristic curves
- Discontinuity of the second derivative of the dependent variable is allowed, but no discontinuity of the first derivative

The differentials of ϕ_x and ϕ_y which represent changes from x,y to $x + dx$, $y + dy$ along characteristics are:

$$d\phi_x = \frac{\partial \phi_x}{\partial x} dx + \frac{\partial \phi_x}{\partial y} dy = \phi_{xx} dx + \phi_{xy} dy \tag{2-146}$$

$$d\phi_y = \frac{\partial \phi_y}{\partial x} dx + \frac{\partial \phi_y}{\partial y} dy = \phi_{yx} dx + \phi_{yy} dy. \tag{2-147}$$

Express (2-145) as

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = H \tag{2-148}$$

with:

$$H = -(D\phi_x + E\phi_y + F\phi + G). \tag{2-149}$$

Assume (2-148) is linear. Solve (2-148) with (2-146) and (2-147) for second derivatives of ϕ:

$$\begin{aligned} A\phi_{xx} + B\phi_{xy} + C\phi_{yy} &= H \\ dx\phi_{xx} + dy\phi_{xy} \qquad &= d\phi_x \\ dx\phi_{xy} + dy\phi_{yy} &= d\phi_y \end{aligned} \tag{2-150}$$

or

$$\begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} \phi_{xx} \\ \phi_{xy} \\ \phi_{yy} \end{bmatrix} = \begin{bmatrix} H \\ d\phi_x \\ d\phi_y \end{bmatrix} \tag{2-151}$$

and solve for ϕ_{xx}, ϕ_{xy}, ϕ_{yy}. Since second derivatives can be discontinuous on the characteristics, the derivatives are indeterminate and the coefficient matrix would be singular:

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0. \tag{2-152}$$

Expanding:

$$A(dy)^2 - B\,dx\,dy + C(dx)^2 = 0 \tag{2-153}$$

and the slopes of the characteristics curves are found by dividing by $(dx)^2$:

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0. \tag{2-154}$$

Solve for *dy/dx*:

$$\left.\frac{dy}{dx}\right|_{\alpha,\beta} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{2-155}$$

and hence the requirement on $\sqrt{B^2 - 4AC}$ to define the type of the PDE as related to the characteristics of the equation. See the references cited above for more details.

2.9 Requirements for a Complete Problem Formulation

When formulating a mathematical representation of a fluid flow problem, you have to consider carefully both the flowfield model equations and the boundary conditions. An evaluation of the mathematical type of the PDEs that are being solved plays a key role in this. Boundary conditions must be properly specified. Either over- or under-specifying boundary conditions will doom your calculation before you start. A proper formulation requires:

- governing equations
- boundary conditions
- coordinate system specification.

All before computing the first number! If this is done, then the mathematical problem being solved is considered to be *well posed*.

2.10 Exercises

1. Convert the unsteady 3D Euler equations from classical non-conservative form to divergence form.

2. Eqn. (2-70) is an unusual form of the equation of state. It is from viewgraphs defining the equations used in **cfl3d**. Turn in your derivation of this equation. Is there a typo?

3. Show how Eqn. (2-76) can be obtained.

4. Why is Eqn. (2-97) not in divergence form?

5. Show that point source and point vortex singularities are solutions of Laplace's equation in two dimensions.

   Recall that a point source can be expressed as:

$$\phi(x, y) = \frac{q}{4\pi} \ln(x^2 + y^2)$$

   and a point vortex is:

$$\phi(x, y) = \frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{y}{x}\right).$$

6. Consider the point source of problem 2. What is the behavior of the velocity as the distance from the source becomes large? What is the potential function for a point source? How does it behave as the the distance from the source becomes large. Comment from the standpoint of having to satisfy the “infinity” boundary condition in a program for a potential flow solution.

7. Find the classification type of the following equations:

   Laplace: $U_{xx} + U_{yy} = 0.$

   Heat Eqn. : $U_y = \sigma U_{xx}$, σ real

   Wave Eqn.: $U_{xx} = c^2 U_{yy}$, c real.

2.11 References

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