SPAARO EXPERIMENT

PURPOSE: The purpose of this experiment is to demonstrate the concept of feedback control and how it can be used to change the dynamic properties of an aircraft. It will also demonstrate that all important aspects of aircraft dynamics must be properly modeled in order to accurately predict the behavior of the closed-loop system.

OVERVIEW: A (non-wind-tunnel) model of a small UAV called SPAARO[1] has been modified so that it can be mounted in the open-jet wind tunnel in such a manner that it is free to rotate (within limits) about the pitch axis. At the pivot point is a potentiometer that can be used to generate a voltage proportional to the angle-of-attack. Hence we can obtain a "signal" that essentially is a measure of the angle-of-attack. The model is also equipped with a servo-motor attached to the elevator. The servo-motor deflects the elevator by an amount that depends upon a control signal that is supplied to it. Thus, one can sense the change in angle-of-attack, by measuring the associated potentiometer voltage, generate a control signal as a function of the sensed angle of attack, and pass this control signal to the servo-motor to deflect the elevator. The objective is design this "feedback control law" such that the elevator deflection will generate a pitch moment that opposes the change in angle-of-attack. In this set-up we use LabVIEW to measure the angle-of-attack signal and to generate the control signal for the servo-motor.

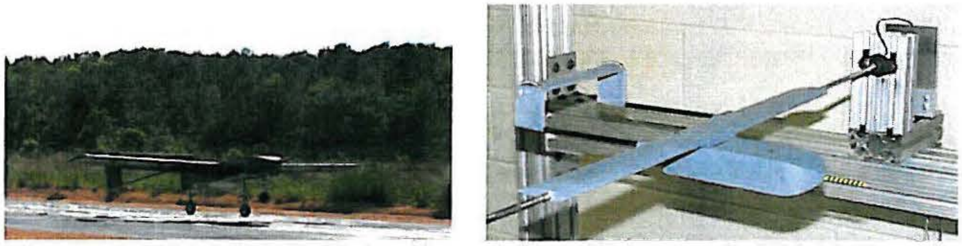

Figure 1. SPAARO (left) landing at Fort Pickett and SPAARO scale model (right) installed in open-jet wind tunnel.

The idea that we are looking at is that by changing the ratio of the control signal (for the servomotor) to the measurement signal (from the angle-of-attack sensor), we can change the dynamic response of the aircraft. This ratio is called the "control gain." If we express the two signals in degrees, for example, the control gain would give degrees of elevator deflection per degree of change in angle-of-attack. Here we deal with voltages, so some of the physical meaning is lost. However, think of it as "elevator angle change per angle-of-attack change."

If the control gain is set to zero, then the elevator remains fixed regardless of the angle-of-attack. In this case the aircraft will undergo its "natural motion" in response to a brief disturbance. If the control gain is increased, the elevator will deflect in proportion to the angle-of-attack. If the gain is increased further, the elevator will deflect more for a given change in angle-of-attack. We will start by examining the natural motion (dynamic response) of the aircraft, i.e., the response with the control gain set to zero. We will then examine the effect of changing the gain on the aircraft dynamic response. Before we do that, however, we should be able to predict what will

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[1] Small Platform for Autonomous Aerial Research Operations

happen as we increase the gain. To do this, we need to look at some governing equations. We will need these equations, in any case, to understand what we are observing in the wind tunnel.

THEORY:

Second Order Dynamic Model: The aircraft model is mounted in the wind tunnel in such a manner that it is free to rotate about the pitch axis. In this configuration, the pitch angle and the angle-of-attack are equivalent. The pitch equation of motion for the "pinned" aircraft is

$$I_y \ddot{\theta} = M, \tag{1}$$

where I_y is the pitch moment of inertia, θ is the pitch angle (or, equivalently, the angle-of-attack), and M is the aerodynamic moment.

If we linearize the mathematical model of the pitch moment M, then equation (1) can be written

$$\Delta\ddot{\theta} = \frac{M_q}{I_y}\Delta\dot{\theta} + \frac{M_\alpha}{I_y}\Delta\theta + \frac{M_{\delta e}}{I_y}\Delta\delta e \tag{2}$$

where $M_{(.)}$ represents the dimensional derivative of M with respect to (.). The term $\Delta\theta$ represents the deviation of the pitch angle from its reference value and $\Delta\delta e$ represents the elevator deflection from its reference value. Equation (2) is a second order, linear time-invariant ordinary differential equation. For typical parameter values, it is an underdamped system. We therefore make the following (implicit) definitions

$$\begin{aligned} \Delta\ddot{\theta} - \frac{M_q}{I_y}\Delta\dot{\theta} - \frac{M_\alpha}{I_y}\Delta\theta &= \frac{M_{\delta e}}{I_y}\Delta\delta e \\ \Delta\ddot{\theta} + 2\zeta\omega_n\Delta\dot{\theta} + \omega_n^2\Delta\theta &= \omega_n^2\Delta\theta_c \end{aligned} \tag{3}$$

In the latter equation, ζ is the damping ratio, ω_n is the undamped natural frequency, and $\Delta\theta_c$ is the commanded change in the pitch angle that corresponds to a given elevator input. Uncontrolled motion (i.e., the "natural" dynamic response) occurs when $\Delta\theta_c = 0$ (i.e., when the elevator is fixed in place). Defining $a = 2\zeta\omega_n$ and $b = \omega_n{}^2$ we have

$$\Delta\ddot{\theta} + a\Delta\dot{\theta} + b\Delta\theta = b\Delta\theta_c$$

Setting $\Delta\theta_c = 0$, the characteristic equation for the system above is

$$\lambda^2 + a\lambda + b = 0 \tag{4}$$

Its solution, assuming that $0 < \zeta < 1$, is

$$\lambda_{1,2} = n \pm i\omega \tag{5}$$

where

$$\begin{aligned} n &= -\zeta\omega_n \\ \omega &= \omega_n\sqrt{1-\zeta^2} \end{aligned} \tag{6}$$

For lightly damped systems ($\zeta < 0.1$), the undamped (ω_n) and damped (ω) natural frequencies are nearly the same. For highly damped systems, however, the two frequencies can be quite different.

A necessary and sufficient condition for dynamic stability of the second order system (3) is that $n < 0$. Alternatively, we can determine the dynamic stability of the second order system directly from the coefficients in the characteristic equation (4). **The *necessary and sufficient* conditions for stability are that $a > 0$ and $b > 0$.**

The solution to the uncontrolled problem (assuming $0 < \zeta < 1$) is

$$\Delta\theta(t) = e^{nt}\left(A\cos\omega t + B\sin\omega t\right) \tag{7}$$

where A and B are determined from initial conditions:

$$\begin{aligned} A &= \Delta\theta(0) \\ B &= \frac{1}{\omega}\left(\Delta\dot{\theta}(0) + n\Delta\theta(0)\right) \end{aligned} \tag{8}$$

If the system is not stable ($n > 0$) or if the damping and frequency do not have the desired values, then feedback control can be used to improve the dynamic properties of the system. The simplest example of feedback is called proportional (P) control. In this case, the control signal is in direct proportion to the measurement. For our experiment, this means the elevator is deflected in direct proportion to the angle-of-attack. If we replace the elevator deflection in equation (2) with $K\Delta\theta$, then the new equation governing the vehicle motion would be:

$$\Delta\ddot{\theta} - \frac{M_q}{I_y}\Delta\dot{\theta} - \left(\frac{M_\alpha}{I_y} + \frac{M_{\delta e}}{I_y}K\right)\Delta\theta = 0 \tag{9}$$

Here we see that only one coefficient (b) is changed. We can also note that if we pick K to have the correct sign, then ***no matter how big we make K, the system will not go unstable!***

A proportional-plus-derivative (PD) control signal

$$\Delta\delta e = K_1\Delta\theta + K_2\Delta\dot{\theta}$$

would allow one to change both coefficients (a and b):

$$\Delta\ddot{\theta} - \left(\frac{M_q}{I_y}\Delta\dot{\theta} + \frac{M_{\delta e}}{I_y}K_2\right) - \left(\frac{M_\alpha}{I_y} + \frac{M_{\delta e}}{I_y}K_1\right)\Delta\theta = 0 \tag{10}$$

Again, if we select K_1 and K_2 to have the correct signs, then ***no matter how big we make the gains, the system will not go unstable (*** $a > 0$ **and** $b > 0$***)!***

For the second order system, we can determine the damping ratio and the undamped natural frequency in terms of the coefficients *a* and *b*:

$$\begin{aligned} \omega_n &= \sqrt{b} \\ \zeta &= \frac{a}{2\sqrt{b}} \end{aligned} \tag{11}$$

In this experiment we have only a simple proportional feedback control law; the elevator deflection is proportional to the pitch angle, so only the coefficient *b* is changed. If we pick the sign of the gain correctly, *b* will be positive and will increase in value as the gain is increased. As a result, as the gain is increased, we expect the undamped natural frequency to increase and the damping ratio to decrease. Thus, as the gain is increased, the observed motion in response to some initial offset in the pitch angle would be higher frequency oscillations with less damping (more oscillations before damping to zero). However, regardless how high the gain becomes, the second order system would not become unstable. As you will see in the experiment, the aircraft behaves as expected for small gains but, at higher gains, the system becomes undamped and, ultimately, ***unstable*** as the gain is increased further. What's happening? It is clear that our model is not sufficiently accurate.

Third Order Model: We can now consider a more detailed mathematical model of the vehicle. Typically, aircraft control surfaces have "dynamics" associated with them; they cannot respond instantaneously to position commands. A common representation for this behavior is to define a first order actuator model:

$$\Delta\dot{\delta e} + \frac{1}{\tau}\Delta\delta e = \frac{1}{\tau}\Delta\delta e_c \tag{12}$$

where $\Delta\delta e$ is the actual control deflection, $\Delta\delta e_c$ is the commanded deflection, and τ is the ***time constant*** for the first order system. Under these circumstances the ***proportional*** feedback control is of the form $\Delta\delta e_c = K\,\Delta\theta$. This servo-actuator command is then integrated in equation (12) and the output ($\Delta\delta e$) is injected into the second order system (2). Thus, the complete system becomes third order. The associated characteristic equation takes the form:

$$\lambda^3 + a'\lambda^2 + b'\lambda + c' = 0 \tag{13}$$

From Routh-Hurwitz stability theory (Routh's criteria), the ***necessary and sufficient conditions for this system to be stable are that***

$$a' > 0, \quad b' > 0, \quad c' > 0, \quad \text{and} \quad a'b' - c' > 0$$

Note again that we do not have to solve explicitly for the roots of the characteristic equation (i.e., the characteristic values) to determine whether the system is stable. If we did solve for the characteristic values, then we would require that the real part of each characteristic value be strictly negative. Routh's criteria provide necessary and sufficient conditions for this to be true.

EXPERIMENT:

Unfortunately we don't have all the information that we need to actually substitute values into the governing equations of motion. We lack the aerodynamic coefficients, and we don't know the model moment of inertia. However, by doing some simple tests, we can determine the coefficients of the characteristic equation with the control fixed (i.e., with $K = 0$). We can put the model in the airstream and allow it to reach an equilibrium state. Then we can disturb it by changing the angle-of-attack (e.g., by pushing the nose up) and then letting go. By observing a graph of pitch angle versus time, we can measure the period (the time between two zero crossings, moving in the same direction, or the time from one peak to the next). Further if we measure the height of one peak and compare it with the height of the previous peak, we can determine the damping ratio. Consequently, by observing the graph of pitch angle versus time, we can determine the frequency and damping for the uncontrolled system.

Next, we can set the gain to some small nonzero value and repeat the experiment: let the aircraft reach equilibrium in the airstream, disturb the angle-of-attack, and observe the response. The frequency and damping should be different. However, if our original model that neglected the elevator dynamics was accurate, we should be able to increase the gain to arbitrarily large values and the aircraft would remain stable. It would oscillate at higher and higher frequencies (the coefficient "b" and hence ω_n is getting bigger) but the damping ratio would decrease (the coefficient "a" and hence $\zeta\omega_n$ remains constant, so ζ must decrease) . We would therefore expect oscillations at a higher frequency and which take longer and longer to damp out. This is not the case, in reality. By increasing the magnitude of the proportional gain we see that the frequency increases and the damping decreases until the system actually goes unstable! Although this behavior is not predicted by the second order model (2), it is predicted by the third order model that includes the elevator dynamics (12). So let's see if we can duplicate this behavior in the experiment and provide the corresponding analysis.

CONTROL OFF: FREE RESPONSE

Although we cannot measure the aerodynamic characteristics of this model, we can get some information about the coefficients in the equations by observing the dynamic behavior of the model as it responds to an initial displacement away from the equilibrium pitch angle. That is, we start by doing some ***parameter identification*** of the model. All measurements will be made using the software LabVIEW. A special program ("virtual instrument" or "VI") was made for this lab so that clicking on the appropriate icon on the screen will initiate data acquisition or perform other desired options. Note that, since we cannot accurately measure angles, we will measure the voltages associated with these angles; all signals will be measured in units of volts.

1. Set the speed of the wind tunnel to ***less than*** one inch of water -- a tunnel setting of 300 RPM works well. Record the tunnel speed, as indicated by the static pressure gauge. (Note that the model used in this experiment is not a wind tunnel model. It is not designed to withstand the loads that a wind tunnel model can withstand. Hence, we keep the speed low.)

2. Start the LabVIEW program titled "SPAARO LAB." You will be prompted to enter a filename for any data that you may wish to record.

1) Set Data Saving to ON
2) Run VI
3) Designate file name and extention (eg, "mydata.dat")
4) Designate time to record
5) Run Tunnel at 500 RPM
6) Tare zero-AoA votage
7) Select "Record Data"

3. Turn feedback control ***off*** by setting the value of the gain to zero. (The gain adjustment is located under the heading "GAIN CONTROL" on the LabVIEW screen. You can turn the knob to zero using the mouse cursor or you can enter the value "0" in the window beside the knob.) Deflect the aircraft in pitch by lifting the nose a small amount and letting go. The resulting motion is the response to a nonzero initial condition (in this case, a displacement). Because the elevator remains fixed in place, this response is also called the "stick-fixed response."

4. Record a time-history of the potentiometer voltage that measures pitch angle. First, designate the length of time you wish to record data and make sure the switch labeled "Data Saving" is toggled to the "ON" position and then click the button labeled "Reset Time and Record Data." It might take a couple of practice runs to get the timing coordinated with the disturbance so that you don't waste a lot of data points on uninteresting data. You should record data for at least five trials at each gain setting of interest (starting with the gain at zero).

5. During one of several trials, you can record the data on a graph on the computer screen. By selecting the "Stop" button, you can stop the recording and "freeze" the image on the screen. It is best to do this after only two or three oscillation cycles. Viewing the graph on the screen, estimate the period, the damped frequency, and the damping ratio. Use the peak-to-peak ratio in the following logarithmic decrement relationship to estimate the damping ratio.

$$\ln \frac{\Delta\theta_1}{\Delta\theta_2} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Note that this relationship can be derived directly from equation (7). Also by visualizing an envelope of the trace, estimate the time to half amplitude from the relationship

$$t_{\text{half}} \approx \frac{0.69}{\zeta\omega_n}$$

Again, this relationship can be derived from equation (7). You will later use these same procedures with your recorded data to obtain more accurate (averaged) results.

CONTROL ON: CLOSED-LOOP RESPONSE

1. Set the speed of the wind tunnel to the same value as used previously.
2. With control on, set the GAIN as low as it can go (zero would be nice) and verify that, when disturbed in pitch, the aircraft behaves as it did before. Note that the gain is simply a number associated with the LabVIEW code and the associated electronics. While it is related to the gain K described in the Theory section (i.e., the ratio of commanded elevator angle to measured pitch angle), the experimental implementation obscures this relationship. The qualitative behavior of the system in response to changes in the GAIN, however, should match theoretical predictions.
3. Increase the GAIN slightly (say, to "0.1") and record data as was done previously. Again, you should take several repeat runs at each value of the GAIN that you select. Pay close attention to any change in the frequency that is observed and the number of oscillations to half amplitude. Estimate how the undamped natural frequency and the damping ratio change with increasing gain. You can make these estimates from the graph on the computer screen during the lab and then repeat them later using the more accurate, recorded data for your write-up.
4. Continue increasing the GAIN and observing the response until you observe the aircraft to behave in an *unstable* manner! Record the value of the GAIN which leads to instability. Note that you may have to repeat this part several times and take an average to accurately estimate the critical value of the GAIN.
5. For choices of the GAIN close to the critical value, when the aircraft is oscillating rapidly, you will be able to observe that there is a time lag between the aircraft displacement in pitch and the elevator deflection. This time lag suggests actuator "dynamics." Observe and estimate this time delay between the elevator and pitch motion; that is, estimate the value of τ in equation (12). Note that τ is not the actual time delay, but is related to it. (Recall from your vibrations course that a stable, first order system subject to a step input will achieve 95% of its steady-state value within 3 time constants and 98% of its steady-state value within 4 time constants.)

Write-up and Calculations: The data that should be collected from this experiment are the recordings of the time history of the oscillations initiated by displacing the model a small amount and letting it go and the value of the associated gain. You should record several (say 5) runs for each gain setting, starting with the gain equal to zero and incrementing by small amounts until instability is observed. You will obtain voltage histories of $\Delta\theta$. Each of these histories should be plotted in Matlab. State your procedure explicitly in your lab write-up.

For each gain setting, you will have five plots. ***You should treat these separately, determine the required information from each, and then average the results.*** (We normally would average the data and then calculate the results, but here we have no control over the initial state, so each curve is independent of the other.)

1. For each time trace of $\Delta\theta$, by measuring peak-to-peak values and times, determine the period (seconds), the damped frequency (Hz), the damping ratio, the time to half amplitude (seconds), the undamped frequency, and the number of cycles to half amplitude. Average the results of these calculations for each gain setting so that, for each gain setting, you have one set of values.

   Note: A more sophisticated way of determining these values is to make use of the known solution given in equation (7). The basic idea is to select the values of a and b (or equivalently ζ and ω_n) so that the sum of the squares of the errors between the measured and predicted time histories is minimized. This procedure is known as "least squares optimization."

2. Since the aircraft has natural rate damping, only proportional control is used here. The elevator deflects in direct proportion to the measured pitch angle. Remember, the actual number used by the computer to define the gain is not *really* the ratio of control angle to pitch angle. From the data calculated in Part 1 make some plots of how the undamped frequency, damping ratio, damped natural period, and time to half amplitude vary with gain.

3. At each gain value, there is a corresponding damping ratio and undamped natural frequency. Associated with each set are the roots of the characteristic equation; see equations (5) and (6). Plot the locus of the roots in the complex plane (real part on the x axis and the imaginary part on the y axis) as the gain increases from zero up to the critical value that leads to instability. The assumption here is that the system is behaving like a second order system, so we are just matching the observed motion (characterized by damping and frequency) with the characteristic values that would give rise to that motion.

   We now need to look at how the theory predicts what we observe. If we assume a second order system then, according to the necessary and sufficient conditions, the coefficients in the characteristic equation (a and b) must be positive. The governing equation for our experiment is equation (9). We know, from the aircraft stability and control course, that M_q, M_α, and $M_{\delta e}$ are all negative for a statically stable aircraft.

4. In light of this information discuss the following:

   a. What must the sign of the gain K be so that the system will not go unstable regardless how large the value gets. This will indicate which direction the elevator will be deflected if the aircraft pitches up (or down): $\Delta\delta e = K\,\Delta\theta$. (Again, note that the gain in LabVIEW is not the same gain you are discussing here; its sign may not match what you expect.)

b. Verify from your knowledge of sign conventions that the result you found in Part 4a makes sense.

c. Verify from the necessary and sufficient conditions for dynamic stability (i.e., that the coefficients in the second order characteristic equation must be positive), that if the sign of the gain is correct, the system can never go unstable.

d. Discuss the case in which the sign of the gain is opposite to the "correct" sign. Show that there is a range of such gains for which the system remains stable and compute this range analytically from the governing dynamic equation. Indicate how the damping and frequency change as the magnitude of the gain is increased.

5. To examine the characteristics of the third order system, we need to:

   a. Combine equations (2) and (12) into a single, third order equation in $\Delta\theta$ with the terms containing $\Delta\theta$ and its derivatives on the left, and the term containing $\Delta\delta e$ on the right.

   b. Assume a proportional control $\Delta\delta e = K\,\Delta\theta$, and write the resulting third order equation in the form:

$$\Delta\dddot{\theta} + a'\Delta\ddot{\theta} + b'\Delta\dot{\theta} + c' = 0$$

6. Assume that the gain has the correct sign (as established in Part 4a). Routh's criteria say that all the coefficients must be positive and, in addition for a cubic polynomial, that:

$$a'b' - c' > 0$$

   a. Assume the variables that appear in the coefficients of the cubic polynomial have the same signs as we assumed for the quadratic equation. In addition, τ is assumed positive. Under all these assumptions, determine if the characteristic equation for the third order system can yield unstable roots (roots with positive real parts) if the magnitude of the gain is increased without limit.

   b. From the results in Part 6a, determine the critical value of K that separates stable and unstable behavior in terms of the aerodynamic parameters, moment of inertia, and the time constant τ.

7. Summarize your analysis and indicate how it shows that, assuming the proper sign of the gain K, a second order system will never go unstable while a third order system will *always* go unstable if the gain is high enough. Discuss how your experimental observations confirm this analysis.

8. Finally, we can use the observations we made to estimate certain properties of the model. With the gain set to zero, we can directly estimate M_q/I_y and M_α/I_y (the coefficients in the uncontrolled equation). There is no way to estimate $M_{\delta e}/I_y$. We can estimate τ by observing the elevator movement as we manually oscillate the aircraft in pitch and watch the elevator. Unfortunately, we don't have the elevator instrumented.

There is one additional bit of information that we can get from all of this. Routh's criteria also tell us that, at the point where the system goes unstable, the damping goes to zero and the undamped frequency is

$$\omega = \sqrt{\frac{c'}{a'}}$$

Since we can measure the frequency at this point, we can relate the gain to all of the above parameters. Unfortunately, since we don't know $M_{\delta e}/I_y$ we can't relate the true gain (elevator angle per pitch angle) to the electronic gain (counts on the LabVIEW screen).

a. Write down the estimates of the two aerodynamic parameters M_q/I_y and M_α/I_y and your estimate of τ and how you obtained it.
b. Write down the frequency of oscillation of the vehicle when it starts to go unstable.
c. Suggest an experiment that we could do to determine some of the missing data so that we could find:
    i. The actual gain at which the vehicle goes unstable (i.e., the mechanical gain).
    ii. The term $M_{\delta e}/I_y$.