$$V_E' = \frac{1}{\rho_\infty}\frac{d}{dx}\left[\rho_\infty U_\infty(\delta_{0.99} - \delta^*)\right] \qquad \text{(P4.5)}$$

for a boundary layer on a plane surface. Compare this with the result from equation (P4.1) of problem 4.8 for a boundary layer.

4.11 Using equation (P4.5) of problem 4.10, derive a relation between V_E'/U_∞ and $C_f/2$ for a zero pressure gradient boundary layer flow with no wall injection or suction. Use equation (P4.2) of problem 4.8 and the relation $\frac{1}{\delta}\frac{d\delta}{dx} = \frac{1}{\theta}\frac{d\theta}{dx}$

4.12 For an axisymmetric turbulent boundary layer on a body of revolution show that

$$V_E' = \frac{1}{\rho_\infty R_\infty}\frac{d}{dx}\left[\rho_\infty U_\infty R_\infty(\delta^r_{0.99} - \delta^{*r})\right]$$

where

$$\delta^r_{0.99} = \int_0^{\delta_{0.99}} \frac{r}{R_\infty}\,dy \,, \qquad \delta^{*r} = \int_0^{\delta_{0.99}} \left(1 - \frac{U}{U_\infty}\right)\frac{r}{R_\infty}\,dy \,,$$

and R is the radius of curvature at the outer edge of the boundary layer.

Chapter 5

Some Fully-Constrained Turbulent Flows

As defined in Section I.2, turbulent fluid completely occupies the flow cross-section in fully-constrained turbulent flow. No turbulent-non-turbulent fluid interface exists so there is no influence of entrainment on the fluid motion. The constraining boundaries govern the fluid motion. During the transition process, however, we find slugs of turbulent fluid preceded and followed by laminar fluid.

The classical example of fully-constrained turbulent flow is fully-developed (no streamwise change in the mean velocity distribution) turbulent flow in a smooth circular pipe. The "law of the wall" for smooth walls, described in Section 3.1, describes the flow near the pipe surface. Large eddy motions and the small shearing stress near the pipe centerline produce a velocity profile slightly different from that given by the law of the wall. In terms of the largest scale of fluid motion, the pipe radius, the flow near the pipe centerline can be described by a "universal velocity defect" relation for high Reynolds numbers. At low Reynolds numbers, when the viscous terms become important outside the viscous sublayer, the constants in the logarithmic law of the wall become Reynolds number dependent.

Similar relations describe fully-developed flow in rough and non-circular cross-section pipes. Flows in non-circular pipes also contain secondary flows as do flows in curved conduits. Turbulent non-Newtonian pipe flow can be described by analogous parameters. Flows in variable cross-section conduits are not fully-developed but contain many characteristics of that condition.

5.1 Fully-Developed Flow in a Smooth Circular Pipe

When fluid enters a pipe of constant radius, there is some distance downstream of the entrance (entrance length) before the velocity profile ceases to vary in the flow direction, i.e. becomes fully-developed. Along this length a boundary layer (half-constrained flow) grows toward the pipe center. If the entering flow is disturbance free, a laminar boundary layer grows until the onset of transition to turbulence, being followed by a developing turbulent boundary layer. When the entering flow is turbulent, the fully-developed condition is approached in a shorter length. We can use the theory of Chapter 7 to predict the entrance length in the former case. Latzko (1921) suggested that the entrance length L_e for the latter case is given by

$$\frac{L_e}{D} = 0.693\, Re_D^{1/4} \tag{5.1}$$

where D is the pipe diameter. The analytical results of Deissler (1953) are in agreement with this equation.

For the fully-developed condition, the governing momentum equation reduces to a balance of pressure and shearing forces, since there is no streamwise variation of flow momentum or Reynolds normal stresses:

$$\tau = \left(-\frac{dP_w}{dx}\right)\frac{r}{2} \tag{5.2a}$$

and

$$\frac{\tau}{\tau_w} = \frac{2r}{D} \tag{5.2b}$$

where r is the distance from the centerline, the shear stress being τ .

Equations (5.2) and (1.26) can be combined to produce

$$\tau = \rho\left(-\overline{uv} + \nu\frac{\partial U}{\partial y}\right) = \tau_w\left(1 - \frac{2y}{D}\right) \tag{5.3}$$

since y = D/2 - r. Equation (5.2b) holds for any fully-developed pipe flow, making equation (5.3) independent of the Reynolds number.

At high Reynolds numbers, the viscous term in equation (5.3) is relatively unimportant in flow outside the viscous sublayer while the shear stress is essentially the wall value ($2y/D << 1$). Under these conditions, a regionally self-preserving wall flow occurs as described in Section 3.1 and the logarithmic equation (3.7) applies.

$$U^+ = \frac{1}{\kappa}\ln|y^+| + A \tag{3.7}$$

Figure 3.1a illustrates the applicability of the law of the wall for smooth walls, described in detail in Chapter 3. Several values for A have been proposed from the survey of much data: Schlichting (1968), A = 5.5; Patel and Head (1969), A = 5.45; and Van Driest (1956), A = 5.2. Judging from the best data fit near the wall, the value A = 5.2 is recommended. Patel and Head suggest that $\kappa = 0.418$ should be used, although the bulk of high Reynolds number data indicate that $\kappa = 0.40$.

Figure 3.1a indicates that at high Reynolds numbers and large y/R, the mean velocity profile departs from the logarithmic law of the wall. The largest size eddy motions in the pipe are of the order of the pipe radius. These eddies occupy and dominate the flow in the center of the pipe, so it is natural to non-dimensionalize the flow distribution on the pipe radius R. We can add a correction function h(y/R) to the right side of equation (3.7) to account for the effects of large eddies

$$U^+ = \frac{1}{K}\ln|y^+| + A + h(y/R) \qquad (5.4)$$

An equation entirely in terms of y/R can be obtained by subtracting equation (5.4) from equation (5.4) evaluated at the pipe centerline ($U = U_{MAX}$, y/R = 1)

$$\frac{U_{max} - U}{U_\tau} = \frac{-1}{K}\ln\left|\frac{y}{R}\right| + h(1) - h(y/R) \qquad (5.5)$$

Equation (5.5) is known as "the velocity defect representation" for the large eddy dominated center portion of the pipe. Note that equation (5.5) does not describe the dynamic sublayer flow since the dominating eddies there scale on ν / U_τ . Figure 5.1 shows the good correlation in the velocity defect representation of the data of Figure 3.1 at large y/R.

Hinze (1959) presented a tentative empirical curve for the correction function h(y/R), shown in Figure 5.2. A reasonably good and simple fit for this correction function is given by the equation

$$h(y/R) = \frac{1}{2}\left[1 - \cos\left(\frac{5\pi y}{4R}\right)\right] \qquad (5.6)$$

It is clear that much of the turbulent energy contained in the center of the pipe originates in the wall flow. (Section 3.3 is devoted to discussion of this wall flow turbulence structure. Section 2.5 describes the behavior of the small scale eddies away from the wall). This turbulent energy is transported mainly by the large eddies of scale R from the pipe wall toward the centerline. Equation (5.7a) is the turbulent energy equation in cylindrical co-ordinates (x, r, ϕ) presented by Laufer (1953) for fully-developed pipe flow

$$\overline{uv}\frac{dU}{dr} + \frac{1}{r}\frac{d}{dr}\left[r\overline{v\left(\frac{q^2}{2}+\frac{p}{\rho}\right)}\right] - \frac{\nu}{r}\frac{d}{dr}\left[r\frac{d}{dr}(\overline{q^2}/2)\right] - \frac{\nu}{r^2}\left(4\,\overline{w\,\partial v/\partial\phi} - \overline{(v^2 + w^2)}\right) + \varepsilon = 0 \qquad (5.7a)$$

where the dissipation is given as

$$\varepsilon = \nu\left\{\overline{\left(\frac{\partial u}{\partial x}\right)^2} + \overline{\left(\frac{\partial v}{\partial x}\right)^2} + \overline{\left(\frac{\partial w}{\partial x}\right)^2} + \overline{\left(\frac{\partial u}{\partial r}\right)^2} + \overline{\left(\frac{\partial v}{\partial r}\right)^2} + \overline{\left(\frac{\partial w}{\partial r}\right)^2} + \frac{1}{r^2}\left[\overline{\left(\frac{\partial u}{\partial \phi}\right)^2} + \overline{\left(\frac{\partial v}{\partial \phi}\right)^2} + \overline{\left(\frac{\partial w}{\partial \phi}\right)^2}\right]\right\} \qquad (5.7b)$$

The fourth term of equation (5.7a) is generally much smaller than ε and is neglected at high Reynolds numbers, i.e. this term is of order $\nu\,\overline{u^2}/R^2$ while ε is of order $\nu\overline{u^2}/\lambda^2$ where λ is the microscale of turbulence. Thus the ratio of these two terms is λ^2/R^2. Only at the lowest Reynolds numbers can this term contribute much. The third term of equation (5.7a) is also relatively negligible away from the wall at high Reynolds numbers. It too is possibly important at very low Reynolds numbers.

The distributions for the remaining three terms are shown in Figure 5.3 for a high Reynolds number flow. Near the wall the sum of pressure and turbulent diffusion is closely zero, as pointed out in Section 3.3, and turbulent production equals turbulent dissipation. Halfway toward the centerline, the turbulent diffusion supplies the larger portion of turbulent energy which is dissipated. Naturally, at the centerline the turbulent energy production must be zero to satisfy the mean flow symmetry. The turbulent energy containing eddies are also the Reynolds shear stress producing eddies. In view of the discussion in Section 3.4, another way of stating the same thing is that the same horseshoe vortices which convert mean pressure energy into turbulent energy produce the Reynolds shear stress.
Figure 5.4 shows the ratio of the mean shearing stress to mean turbulent

energy to be nearly constant over the midrange of tube radii. Figure 5.5 shows why this ratio drops to zero at the centerline - the shear stress vanishes while there is a residual turbulent energy, solely available for turbulent dissipation. Figures 5.6 and 3.2a show the distribution of the turbulent energy among the three component quantities. Notice that there is still a considerable Reynolds number effect as close to the wall as $y = 0.2R$ $r = 0.8R$ for Reynolds numbers between 5×10^4 and 5×10^5.

Relatively little quantitative data are available to examine low Reynolds number effects on the turbulence flow structure. Clearly, when the Reynolds number is low, no regionally self-similar flow can exist outside the sublayer and equations (5.4) to (5.6) become Reynolds number dependent, as do the terms in the turbulent energy equation. Figure 5.7 shows data of Patel and Head (1969). For $Re_D > 10^4$, equations (5.4) to (5.6) appear to describe the data. At lower Reynolds numbers the large eddy correction function h(y/R) to the logarithmic distribution is weaker, apparently vanishing near the lowest Reynolds number at which the entire flow crosssection is occupied by turbulent fluid. Patel and Head (1969) observed intermittently laminar and turbulent flow below a Reynolds number of about 3000, indicating this value as the upper limit on the range of transition Reynolds numbers.

For $3 \times 10^3 < Re_D < 10^4$, they observed the logarithmic form of the velocity distribution outside the dynamic sublayer with K constant, but with the parameter A a function of Re_D as shown in Figure 5.8. At these low Reynolds numbers, the dynamic sublayer is relatively thick ($0.03 < \frac{\delta_s}{R} < 0.1$), with the shearing stress not nearly constant and the viscous contribution outside the sublayer no longer negligible - violations

of assumptions for deriving equation (3.7). Thus it is expected that much of the flow structure for low Reynolds numbers is substantially Reynolds number dependent. Apparently for $Re_D \approx 10^5$, there is some Reynolds number dependence on the turbulence structure (Figure 5.6) although there is little Reynolds number dependence on the mean velocity profile.

From a practical viewpoint, probably the pressure drop is the most important parameter. It can be calculated from equation (5.2a) when τ_w is known. A friction drag law in terms of the average velocity Reynolds number can be produced by integrating the velocity profile over the pipe area. For high Reynolds numbers, equation (5.4) would be used for U^+ in the relation

$$\frac{\overline{U}}{U_\tau} = 2\int_0^1 U^+(1-y/R)\,d(y/R) \qquad (5.8)$$

where $\overline{U}$ is the average velocity over the flow cross section. Traditionally the pipe friction factor f, defined by

$$f = \frac{2(-dP/dx)D}{\rho \overline{U}^2} \quad \text{or} \quad f = 8\left(\frac{U_\tau}{\overline{U}}\right)^2 \qquad (5.9)$$

is used. Thus for high Reynolds numbers equation (5.8) produces

$$\frac{1}{\sqrt{f}} = 2.035 \log_{10}\left|Re_D\sqrt{f}\right| - 0.842 \qquad (5.10)$$

for A = 5.2. For A = 5.45 the constant 0.842 becomes 0.754. We should note the importance of considering h(y/R) in deriving equation (5.10). Without it the constant 0.842 would become 1.02, indicating relatively poorer agreement with data. Schlichting (1968) suggested that "Prandtl's Universal Law of Friction for Smooth Pipes"

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10}\left|Re_D\sqrt{f}\right| - 0.8 \qquad (5.11)$$

fits the data shown in Figure 5.9, indicating only a slight discrepancy between equation (5.10) and experimental data. Schlichting also notes that equation (5.11) also correlates supersonic friction factor data. Of course, equation (5.4) with h(y/R) given by equation (5.6) does not apply for $Re_D\sqrt{f} \lesssim 3.2$ or $Re_D \lesssim 10^4$, explaining why equation (5.10) does not fit the data well in this range. Furthermore, we neglected the sublayer effect in obtaining equation (5.10), which when considered consists of an additive term $142/Re_D$ on the right side. This latter expression is only valid for $Re_D > 10^4$, but is negligible.

At lower Reynolds numbers, $3 \times 10^3 < Re_D < 10^5$, the Blasius (1913) formula correlates friction factor data

$$f = 0.3164\, Re_D^{-1/4} \qquad (5.12)$$

as shown in Figure 5.9. One could use equation (5.8) to determine f at low Reynolds numbers if h(y/R) and A were known functions of Re_D and the sublayer contribution was properly accounted for.

5.2 Fully-Developed Flow in a Rough Circular Pipe

The major difference between smooth and rough pipe flow is that the law-of-the-wall velocity profile is no longer given by the smooth wall

equation (3.7), but is described by equation (3.64), the rough wall expression. Everything descussed in Section 3.6 applies here. The flow near the pipe center is large eddy dominated as in the smooth wall case and scales on the pipe radius. Thus for small k/R and high Reynolds numbers, the rough circular pipe velocity profile is given by

$$U^+ = \frac{1}{K}\ln\left|y^+\right| + A - \frac{\Delta U}{U_\tau}\left(\frac{kU_\tau}{\nu}, \lambda\right) + h(y/R) \qquad (5.13)$$

where h(y/R) is given by equation (5.6). K and A and given by the smooth wall flow.

Because the flow structure near the pipe center scales on R, no effect of roughness on the non-dimensional flow structure is detected for small k/R when data are plotted in terms of y/R. The velocity defect representation, equation (5.5), holds; if equation (5.13) is subtracted from equation (5.13) evaluated at the centerline, equation (5.5) results. At high Reynolds numbers the flow structure shown in Figures 5.3 - 5.6 is valid. In essence the only effect roughness has on the flow away from the wall is to increase U_τ on which the flow structure is normalized. Near the wall, however, the flow structure is highly dependent on the roughness element geometry.

To derive the drag law for rough pipe flow we can use equation (5.13) in equation (5.8) to produce

$$\frac{1}{\sqrt{f}} = 2.035 \log_{10}\left|Re_D\sqrt{f}\right| - 0.842 - \frac{\Delta U}{U_\tau}\left(\frac{kU_\tau}{\nu}, \lambda\right) \qquad (5.14)$$

Naturally, when $k \to 0$, this equation reduces to equation (5.10). When $kU_\tau/\nu > 70$, the wall flow is in the completely rough regime and equations (3.65) and (3.66) describe the law of the wall independent of $\frac{kU_\tau}{\nu}$

, should be k

$$U^+ = \frac{1}{\kappa} \ln\left|\frac{y}{R}\right| + A - f(\lambda) + h(y/R) \qquad (5.15)$$

check on this. k or R ?

When this equation is used in equation (5.8) to produce "the drag law for fully-rough pipes", we obtain

$$\frac{1}{\sqrt{f}} = 2.035 \log_{10}\left|\frac{R}{k}\right| + 0.693 - 0.354 f(\lambda) \qquad (5.16)$$

which is independent of Reynolds number.

For many years pipe roughness data have been presented in terms of an "equivalent sand grain roughness k_s", defined as that value required to fit the roughness equation for sand grains

$$\frac{1}{\sqrt{f}} = 2.035 \log_{10}\left|\frac{R}{k_s}\right| + 1.75 \qquad (5.17)$$

to experimental friction data (Schlichting, 1968). Equations (5.17) and (5.16) imply that

$$k_s = k \exp\left[\kappa f(\lambda) + 1.2\right] \qquad (5.18)$$

The equivalent sand grain roughness representation generally does not provide an a priori method of determining pipe friction, whereas the discussion

of Section 3.6 does. Figures 5.10 are Moody's diagrams for determining the friction in commercially available pipes in terms of k_s. In this case actual λ and k values are difficult to determine and a correlation of k_s/D for similar pipes is quite useful. Equation (5.17) is in close agreement with Figure 5.10 when the friction factor is independent of Re_D.

The low Reynolds number effects observed for smooth wall flow are not as pronounced in rough wall flow since the Reynolds number is less important a parameter. Only for $Re_D < 10^4$ and very small roughness effects are velocity profiles expected to show low Reynolds number effects. Even then there is a negligible effect on pipe friction since the low Reynolds number effects on smooth pipe friction is small.

5.3 Fully-Developed Flow in Non-Circular Cross-section Pipes

The oldest and simplest way of predicting the pressure drop in a non-circular pipe is through the parameter known as the "hydraulic diameter". The pressure drop and wall friction are related by the force balance equation

$$\left(\frac{-dP}{dx}\right) A = \bar{\tau}_w c = \frac{cf}{8} \rho \bar{U}^2 \qquad (5.19a)$$

or

$$f = \frac{2(-dP/dx)(4A/c)}{\rho \bar{U}^2} \qquad (5.19b)$$

where A is the flow area and c is the wetted perimeter of the pipe. Here $\bar{\tau}_w$ is the wall shear stress averaged over the wetted perimeter. For a circular pipe, $4A/C$ reduces to D and equation (5.19b) reduces to the first of equations (5.9). This grouping of parameters is the hydraulic diameter D_h:

$$D_h \equiv \frac{4A}{c} \tag{5.20}$$

When D_h is used as the diameter in equation (5.11) or equation (5.12) for the friction factor, good agreement with experimental data is obtained, as long as the angle between adjacent sides of the cross-section is not less than about 45^o. Figure 5.11 shows this good agreement for annular, rectangular, and triangular cross-sections.

Patel and Head (1969) found that their 48:1 aspect ratio rectangular pipe friction data could not be correlated using D_h. Recently, Beavers, et al. (1971) obtained friction data in a 35:1 aspect ratio rectangular channel that were well correlated by the hydraulic diameter concept. This apparent discrepancy serves to show that the hydraulic diameter concept should be used judiciously.

A plausibility argument can be given that will indicate when this concept can be safely used. At high Reynolds numbers, the velocity profile near the center of a duct is rather flat with $U \approx \overline{U}$ (see problem 5.10). The wall flow is in a rather thin layer(s) ($y/D_h < 0.05$) around the perimeter of the duct. Most of the turbulent energy production and dissipation and the direct dissipation occurs in this layer and is proportional to $\overline{U}\,\overline{\tau}_w c$. The wall flow structure of section 3.3 should approximately hold for the non-circular ducts when D_h can be used. The energy supplied the wall flow comes from the mean pressure energy given

up during the pressure drop along the duct, this energy being proportional to $(-dP/dx)\,A\,\overline{U}$. When the ratio of these proportionality constants is the same as for a circular pipe

$$\frac{\overline{U}\,\overline{\tau}_w c}{\overline{U}\left(\frac{-dP}{dx}\right)A} = 4 \tag{5.21}$$

the concept of a hydraulic diameter is valid. If the turbulent production and dissipation structure in a duct varies greatly from that for the circular pipe or if the velocity profile is not relatively flat over the duct center, we cannot expect the proportionality constants to be closely the same as for a circular duct with the same D_h. Thus equation (5.21) would not be valid and the hydraulic diameter concept fails.

Figure 5.12 shows typical secondary flows that occur in non-circular ducts. The work of Gessner and Jones (1965) provides an explanation of this phenomenon. In corners there is greater energy dissipation since shearing stresses act on the fluid from the two walls forming the corner. Hence a greater $-dP/dx$ is required in the corner than in the duct center in order to supply the needed energy for dissipation. Because of continuity considerations, no kinetic energy is available for this purpose. The pressure in the duct center is slightly greater than that in the corner, producing a secondary flow from the duct center to the corners. The continuity equation and the geometrical symmetry then require that a return flow be formed along the sides and back to the duct center.

As long as the angle between two adjacent sides is not too small (greater than about 45^0), a sufficiently great secondary flow can be maintained in a corner, supplying the energy needed to maintain a turbulent flow. In this case we would expect the hydraulic diameter concept to be approximately valid, as the data in Figure 5.11 indicate. Eckert and Irvine (1956) found that for a small angle between the sides of a corner (12^0), the flow in the corner remained laminar even for hydraulic diameter Reynolds numbers of 8000. Naturally the hydraulic diameter concept fails in this case.

Aranovitch (1971) proposed an approximate method for predicting velocity profiles in non-circular pipes. He assumed that the flow away from the walls in the so-called "turbulent core" was governed by

$$\frac{1}{\rho}\frac{dP}{dx} = \varepsilon_M\left[\frac{\partial^2 \mathcal{U}}{\partial y^2} + \frac{\partial^2 \mathcal{U}}{\partial z^2}\right] \tag{5.22}$$

where d /dx is given by equation (5.19b), ε_m is the eddy viscosity assumed constant over the core, and y and z are coordinates in the plane of the cross-section. Equation (5.22) is a Poisson's equation, which when ε_m is replaced by ν is the governing equation for fully-developed laminar flow. The solution for the laminar flow in the same cross-section is assumed to provide the peripherial variation of the surface stress in the turbulent case. ε_m is given as an effectively average value

$$\frac{\varepsilon_M}{\nu} = 0.033\, Re_D \sqrt{f/2} \tag{5.23}$$

derived from circular pipe data.

The flow near a wall is given by equation (3.7) with the local value of U_τ. This local value of U_τ is determined by the shear stress peripherial distribution and the peripherial average $\overline{U}_\tau$ for a given hydraulic diameter. Finally, the core velocity profile is shifted by an additive constant so that the average velocity is $\overline{U}$. Fairly good agreement between results from this method and experiments were obtained for the case of flow along nuclear reactor fuel rods.

The above discussion is valid for rough walls, as long as λ and $2k/D_h$ are constant around the wetted perimeter. When these parameters are not constant or k/D_h is large, secondary flows even in circular pipes can be produced (Townes, et al., 1971). In general we cannot rely on the hydraulic diameter concept for these latter cases since the wall flow structure and the secondary flow pattern are uncertain.

5.4 Non-Fully-Developed but Fully-Constrained Flows

In the first three sections of this chapter, we have dealt with fully-developed flows with some of the flow structure details for the circular pipe case. Here we will outline some approaches for approximate solutions to fully-constrained turbulent flows in variable cross-sectional area ducts, ducts with bends, and ducts containing obstructions such as screens.

Sovran and Klomp (1965) presented a general momentum integral equation for internal flows with arbitrary cross-sectional area shape. For flows in which dA/dx is not large, this equation is

$$d(\rho \overline{\mathcal{U}}_M^2 A_\theta) = A_B dP + \overline{\tau}_\omega c\, dx + (A - A_B) dP_{oM} \tag{5.24}$$

where A_B and A_θ are cross-sectional integral parameters defined as

$$A_B = \int_A \left(1 - \frac{U}{U_M}\right) dA \tag{5.25a}$$

$$A_\theta = \int_A \left(1 - \frac{U}{U_M}\right)\left(\frac{U}{U_M}\right) dA \tag{5.25b}$$

dP_{OM} is the change in stagnation pressure along the streamline or streamlines of maximum velocity U_M. A, c, and $\overline{\tau}_w$ are the local cross-sectional area, perimeter, and perimeter-averaged wall stress.

When the internal flow has an inviscid core surrounded by a turbulent wall flow, the loss term dP_{OM} is zero and equation (5.24) applies to boundary layer flows. In that case we would not have a fully-constrained turbulent flow; the flow would be handled as a boundary layer flow (Chapter 7). Equation (5.24) reduces to equation (1.29) for plane wall boundary layers. Looking at another case, the terms containing a change in the velocity U vanish for a fully-developed flow, reducing equation (5.24) to equation (5.19a).

Here we will look further at the case when dP_{OM} is not zero. The definition of the stagnation pressure and the time-averaged equation (1.24) along the maximum velocity streamline in the X direction produce

$$dP_{OM} = d\left(P_M + \frac{1}{2}\rho U_M^2\right) = \left(\frac{\partial \tau}{\partial n} - \frac{\partial}{\partial x}\left(\overline{u^2} - \overline{v^2}\right)\right)_M dx \tag{5.26}$$

where n denotes a normal to this streamline. The Reynolds normal stress terms are negligible within our order of approximation. Thus, a change in stagnation pressure along a streamline is produced by momentum exchanges with adjacent streamlines, the magnitude depending upon the local shearing stress gradient $\partial\tau/\partial n$. From another viewpoint, the shearing stress acting on a local gradient of velocity across the flow stream drains energy from the mean flow, producing turbulence and to a lesser degree direct viscous dissipation. To predict dP_{OM}, one must tie $(\partial\tau/\partial n)_M$ to the internal flow structure. Two possible ways are: (a) equation (1.24) and either a phenomenological model as presented in Section 1.9 and/or the turbulence energy equation (2.58) can be used in a differential scheme; (b) an integral method relating an assumed velocity profile to τ_w and $\partial\tau/\partial n$.

The static pressure is nearly constant over any cross-section of most uncurved fluid streams and all streamlines experience the same pressure change in passing between the same two stations along the flow path. Ignoring losses, then Bernoulli 's equation indicates that the change in velocity required along each streamline will be inversely proportional to the local velocity. In diffusers ($dP/dx > 0$) the greatest reduction in velocity occurs where the local velocity is smallest - velocity differences across the flow will be increased. Accelerating flows have the opposite effect, making the velocity profile more uniform. As suggested by the several phenomenological relations in section (1.9), the increased velocity variation produced by a decelerating flow produces a higher $(\partial\tau/\partial n)_M$ and an increased loss in stagnation pressure. Diffusers are used to recover pressure from a decelerating stream. Their effectiveness is strongly dependent on the inlet flow distribution and the pressure gradient dP/dx. Too rapid a rise in pressure produces a separated or stalled region (See Chapter 7). Sovran and Klomp (1965) present a review of previous diffuser literature. Accelerating flows have smaller velocity variations resulting in less $(\partial\tau/\partial n)_M$ and a smaller loss in stagnation pressure. Contractions are commonly used in wind tunnels to improve velocity profile uniformity and reduce the turbulence intensity level.

The methods of inviscid flow can be used to approximately determine the flow pattern in a duct (Hawthorne, 1965). In the flow through compressors and turbines, round bends and through screens there are large regions in which the flow may be treated as frictionless, although it is non-uniform. Hawthorne (1965) presented three types of approximate solutions to the problem of three-dimensional, inviscid, steady flow:

1.) Flows with small shear and small disturbances from a known flow pattern. The disturbances are irrotational flows. The shear flows upstream and downstream of a wire screen or a row of blades are examples.
2.) Flows with small shear but large disturbances. This flow consists of an irrotational primary flow that convects the vortex filaments that produce a secondary flow. These solutions have been applied to flow in bends, cascades and around cylinders and spheres. However, this theory does not predict the distortion of the surfaces of constant stagnation pressure (Bernoulli surfaces) found in practice.
3.) Flows with large shear and small disturbances. The disturbance is a rotational flow superimposed on a primary rotational flow. Flow past slender bodies and flow through screens and blade rows are examples when the shear is large.

The applicable equations, details of the method, and bibliography of solutions and experimental results are given by Hawthorne (1965).

5.5 Turbulent Couette Flow

The flow between parallel planes in relative motion is commonly named Couette flow after M. Couette who studied the flow between rotating concentric cylinders. In the fully-developed condition the velocity mid-way between the two planes U_{CL} relative to a wall is half the difference between the velocities of the planes. The velocity profile of motion relative to the nearer wall is identical for the two walls: identical values of velocities relative to a wall produce the same internal flow structure near each wall.

At high Reynolds numbers we can expect the logarithmic equation (3.7) to describe the flow outside the dynamic sublayer. Equation (3.3b) describes the viscous sublayer mean velocity profile. Hence if 2h is the gap width and

$$\frac{C_f}{2} = \frac{\tau_w}{\rho U_{CL}^2} \quad , \quad Re_h = \frac{U_{CL} h}{\nu}$$

then equation (3.7) evaluated at $y = h$ with $\kappa = 0.4$ and $A = 5.2$ implies

$$\frac{1}{\sqrt{\frac{C_f}{2}}} = 2.5 \ln\left[Re_h \sqrt{\frac{C_f}{2}}\right] + 5.2 \tag{5.27}$$

Figure 5.13 shows that equation (5.27) is in very good agreement with experimental data for $Re_h > 5000$. For Re_h below 5000, low Reynolds number effects are significant. Since the hydraulic diameter Reynolds number Re_D is twice Re_h in this case, the largest Re_D at which low Reynolds number effects become important is about 10^4. This value is in good agreement with the $Re_D = 10^4$ result obtained for circular pipe flow.

At lower Reynolds numbers, a variation of A in equation (3.7) similar to but not as strong as that shown in Figure 5.8 would improve the agreement between equation (5.27) and data. The dynamic sublayer is relatively thick; it completely occupies the entire half width of the gap at $Re_h \approx 10^3 (h^+ = 50)$. Note that this is near the upper end of the transitional range of Reynolds numbers $400 < Re_h < 800$. This suggests that a fully-developed Couette flow without transitional effects or intermittently turbulent motion cannot be maintained until the dynamic sublayer is completely accommodated. See section 5.7 below.

Burton and Carper (1967) considered the application of Couette flow to annular turbulent film lubrication between a bearing and a moving journal. They noted that transition occurs at lower Reynolds

numbers and the friction at a given Reynolds number is slightly larger than in the plane case. These effects are presumably due to the formation of the secondary flow Taylor vortex cells by centrifugal body forces. High momentum fluid is forced away from the rotating journal to the bearing surface where the momentum is given up as friction.

5.6 The Transport of Scalar Quantities in Pipe Flow

Scalars such as thermal energy and mass species concentration mainly are carried passively by the turbulent fluid, except near the constraining walls where molecular conduction or diffusion is important (Kays, 1966). Here we will restrict our discussion to small temperature differences and concentrations to avoid variable property effects. Although the discussion below is only for temperature, analogous arguments can be given for mass species concentration.

Consider the time-averaged thermal energy equation describing the temperature in a turbulent flow field (derived in problem 1.7)

$$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{k}{\rho c_p}\frac{\partial T}{\partial y} - \overline{vt'}\right) - \frac{\partial}{\partial x}(\overline{ut'}) \tag{5.28}$$

Here we have neglected axial conduction since $\partial T/\partial x$ near the wall is negligibly small in the absence of large wall temperature variations. On the left side of this equation are the mean velocity convective terms. The first term on the right side is the gradient of the molecular conduction and the Reynolds heat flux normal to a heated or cooled pipe wall.

The second term is the streamwise gradient of the streamwise Reynolds heat flux. With exception of the streamwise pressure gradient term, equation (1.19) is the same form as equation (5.28). If the turbulent transport of momentum and thermal energy are by the same mechanism, then with

$$\varepsilon_M = \frac{-\overline{uv}}{\partial \overline{U}/\partial y} \qquad \text{and} \qquad \varepsilon_H = \frac{-\overline{vt}}{\partial T/\partial y}$$

ε_H and ε_M are equal. This hypothesis is known as the <u>Reynolds Analogy</u>. Figure 7. shows data for heat transfer in a pipe and in a boundary layer. Clearly, the ratio of ε_M to ε_H , or the turbulent Prandtl number Pr_t, is less than unity away from a wall. There is yet no widely accepted explanation of the difference in the transport of momentum and thermal energy but the discussion below provides some insight.

Bremhorst and Bullock (1970) measured the spectra of temperature and streamwise velocity fluctuations in fully-developed pipe flow. Near the wall, velocity and temperature spectra are nearly the same except at high wave numbers. The normalized cross-correlation of velocity and temperature fluctuations is near unity, indicating the close relation between velocity and temperature fluctuations near a wall. In view of the wall flow structure presented in Chapter 3, we see that this strong correlation results because the same fluid composing the horseshoe vortex structures also carries thermal energy. The "bursting" frequency spectrum, which lies at low wave numbers, is closely the same

tracing element hypothesis

as that for the temperature fluctuations. At high wave numbers the energy of the velocity fluctuations are passed by the cascade process to higher wave number fluctuations, with vortices of comparable size influencing one another while those of widely different sizes influence one another very little (Section 2.2). Temperature fluctuations need not obey this cascade process. The higher wavenumber temperature fluctuations can be created by the transport of thermal energy between eddies of widely different sizes.

Further away from the wall there is less similarity between velocity and temperature spectra. The normalized cross-correlation of velocity and temperature fluctuations is much less than unity even at low wavenumbers.

Thus velocity and temperature fluctuations near a wall originate in the bursting phenomenon. However, as these vortex structures move out into the core of the pipe, the dissimilar way in which high wave-number components are created produces less overall correlation. Since the thermal energy transport is not restricted by the cascade process, the apparent ε_H should be greater than ε_M and Pr_t should decrease with distance from the wall.

5.7 <u>Transition from Laminar to Turbulent Flow and Relaminarization of Turbulent Flow in Fully-Constrained Conditions</u>

The discussion of section 1.5 introduced the concept that flow instabilities in laminar flow amplify disturbances until the flow breaks up into turbulent vortices downstream. In a circular pipe flow, the distance downstream of the entrance where the flow is totally turbulent depends on the level of the entering disturbances and the pipe Reynolds numbers. For Reynolds numbers less than 2000, the flow remains laminar or decays to a laminar state even in the presence of strong disturbances.

The non-linear stability theory required to extend the linear theory of section 1.5 to the point of predicting turbulent vortices is not yet well developed. However, we can examine the nature of transitional turbulence formation in terms of our discussion in section 3.2 for turbulent wall flows. In the wall region of a turbulent flow, we found that a dynamic viscous sublayer grows until the instantaneous velocity profile becomes unstable to the surrounding turbulent disturbances. Because of the high level of disturbances, a horseshoe vortex quickly forms producing turbulent energy and shearing stress. In analogous fashion, "turbulent spots" are formed in the transition from laminar to turbulent pipe flow.

O. Reynolds (1883) was the first to observe the breakdown of the prevailing fluid pattern at intervals in time and at various locations in a circular pipe. Depending on the surrounding disturbance level, downstream the turbulent spots from these breakdowns may dissipate or they may increase in size, mainly in their lateral and longitudinal dimensions. When a turbulent spot has increased its dimensions so that its width covers the entire pipe perimeter, it completely fills the flow cross section with turbulence at its instantaneous location (Lindgren, 1969). This "turbulent slug" may

or may not maintain itself under the surrounding flow conditions. As turbulent slugs are formed, the probability for each turbulent spot to develop into a slug increases, as does the probability for already formed turbulent slugs to remain and grow.

Figure 5.14 shows hot-wire anemometer signal traces obtained simultaneously at two positions across a pipe flow area. These data indicate that a turbulent slug forms a right circular cylinder whose front and rear faces are each more or less planes perpendicular to the pipe axis. The flow behind and ahead of a turbulent slug is laminar. Rotta (1956) found that this laminar flow possesses a well-developed parabolic velocity profile both upstream and downstream of a turbulent slug. Within a turbulent slug there is a flattened turbulent velocity profile resembling a fully-developed turbulent pipe flow profile. The centerline velocity of laminar flow is double the mean flow velocity $\overline{U}$ while the propagation velocity or celerity of the slugs is often less than $\overline{U}$. Based on a kinetic energy analysis, Lindgren (1969) predicted that the celerity of the slug rear was 0.7 $\overline{U}$, whereas experiments indicate that it decreases exponentially from U at $Re_D = 2000$ to about $0.65\overline{U}$ at $Re_D \approx 7000$. The slug front celerity increases from about $0.9\overline{U}$ at $Re_D \approx 2000$ to about $1.6\overline{U}$ at $Re_D \approx 7000$. Hence to satisfy the continuity requirement, the laminar flow near the centerline must move into the turbulent slug behind it and/or move toward the walls. The laminar flow in front of a slug must move to the centerline and/or be engulfed by the approaching turbulent slug.

could have ? [illegible]

Figure 5.15 shows the intermittency factor γ, the fraction of time the flow cross-section is occupied by turbulent fluid, for various pipe Reynolds numbers as a function of axial distance downstream of the pipe entrance. For a given Reynolds number the intermittency increases downstream since turbulent slugs grow in length by entraining laminar fluid on their upstream and downstream faces. The greater the Reynolds number, the more intense the entrainment mechanism and the greater rate at which the intermittency increases. Note that when the entering flow is relatively disturbance free, as that associated with the data of Figure 5.15, several hundred diameters must be traveled before the flow is entirely turbulent. At the lowest Reynolds numbers shown, the transition process extends over very large distances, the order of thousands of diameters. The dependence of γ on the Reynolds number, inlet conditions, and the downstream location is not yet well defined.

Pressure drop predictions for the transition range of Reynolds numbers are relatively uncertain due to the uncertainty in γ. The average wall shearing stress is given by

$$\overline{\tau}_w = \gamma\, \tau_{w\,\underset{Re_D}{turb}} + (1-\gamma)\, \tau_{w\,\underset{Re_D}{lam}} \qquad (5.29)$$

where $\tau_{w\,turb,\,Re_D}$ is the shearing stress for an entirely turbulent ($\gamma = 1$) fully-developed flow and $\tau_{w\,lam,\,Re_D}$ is the shearing stress for

an entirely laminar ($\gamma = 0$) fully-developed flow, each evaluated at the transitional flow Reynolds number. The justification for equation (5.29) is the fact that the turbulent and laminar portions appear to possess fully-developed turbulent and laminar velocity profiles, respectively.

Figure 5.15 implies that in general no fully-developed (no axial direction variations) transitional pipe flow exists. Only under special circumstances would γ be expected to remain independent of axial position. Thus unlike fully-developed flow, transitional pipe flow contributes a change in momentum term to the momentum equation. Consider a transitional flow where the laminar and turbulent velocity profiles are given, respectively, as

$$\mathcal{U} = \overline{\mathcal{U}}\left[1 - \left(\frac{2r}{D}\right)^2\right] \qquad (5.30a)$$

and

$$\mathcal{U} = \frac{(n+1)(2n+1)}{2n^2}\, \overline{\mathcal{U}} \left(\frac{2y}{D}\right)^{1/n} \qquad (5.30b)$$

(See problem 5.10 for discussion of equation (5.30b).) The rate of change in momentum of fluid as it passes from a laminar to turbulent state is approximately

$$\Delta M = \int_0^R \rho \left[U_{turb}^2 - U_{lam}^2 \right] 2\pi r \, dr$$

$$= \rho U^2 \frac{\pi D^2}{4} \left[\frac{(n+1)(n+2)}{8n^2} - \frac{2}{3} \right]$$

Thus, we see that in addition to wall friction the rate of change of fluid momentum contributes to the pressure change. For a control volume dx long over the inner wall of a pipe, one obtains the balance equation

$$-\left(\frac{dP}{dx}\right) \frac{\pi D^2}{4} dx = \pi D \tau_w dx + \frac{d\gamma}{dx} \Delta M \, dx$$

where τ_w is given by equation (5.29). The pressure drop is

$$f_{tran} = \frac{2D(-dP/dx)}{\rho U^2} = \gamma f_{turb} + (1-\gamma) f_{lam} + 2D \frac{d\gamma}{dx} \left\{ \frac{(n+1)(n+2)}{8n^2} - \frac{2}{3} \right\} \qquad (5.31)$$

where f_{turb} and f_{lam} are the fully-developed turbulent and laminar friction factors for the Reynolds number of the transitional flow. The case n = 7 is commonly used to fit lower Reynolds number turbulent velocity profiles (see problems 5.9 and 5.10). Equation (5.31) is to be preferred over the transitional friction factor data shown in Figure 5.9 since γ and $d\gamma/dx$ are considered.

Relaminarization of a turbulent pipe flow is a relatively newly studied phenomenon. The decay of turbulence is generally associated with the acceleration of fluid, which produces relaminarization in external boundary layers (Chapter 8). Strong heating of a gas in a constant area pipe produces an acceleration that satisfies the continuity requirement for an increasingly less dense gas (Coon and Perkins, 1970; Bankston, 1969). In addition the viscosity increases, producing lower Reynolds numbers. Injection of gas into a fully-developed turbulent flow has produced relaminarization over a short downstream distance where the acceleration was large (Rodi, et al.,1969). Relatively little is known about the flow structure during relaminarization.

Very little is known about transition in pipes of non-circular cross-section. Goldstein (1938), reports that transition in square, rectangular and annular cross-sections occurs at hydraulic diameter Reynolds numbers of about 1900 to 2300. The hydraulic diameter concept should be used tentatively in predicting the transitional flow characteristics in non-circular pipes.